Exercise 1. Evaluate

$$\int_1^{10} 5\sqrt{t} + \frac{2}{t^{1/3}} dt.$$

Solution: By the fundamental theorem of calculus, we know that

$$F(10) - F(1) = \int_1^{10} 5\sqrt{t} + \frac{2}{t^{1/3}} dt$$

where $F(t)$ is an antiderivative of $5\sqrt{t} + \frac{2}{t^{1/3}}$. Using the "anti-power rule", we see that

$$\int \sqrt{t}dt = \int t^{1/2}dt = \frac{1}{\frac{1}{2}+1}t^{1/2+1} = \frac{2}{3}t^{3/2}$$

and

$$\int \frac{1}{t^{1/3}}dt = \int t^{-1/3}dt = \frac{1}{-\frac{1}{3}+1}t^{-1/3+1} = \frac{3}{2}t^{2/3}.$$

So, an antiderivative is given by

$$F(t) = 5\left(\frac{2}{3}t^{3/2}\right) + 2\left(\frac{3}{2}t^{2/3}\right)$$

which implies

$$\int_1^{10} 5\sqrt{t} + \frac{2}{t^{1/3}} dt = \boxed{5\left(\frac{2}{3}(10)^{3/2}\right) + 2\left(\frac{3}{2}(10)^{2/3}\right) - 5\left(\frac{2}{3}(1)^{3/2}\right) - 2\left(\frac{3}{2}(1)^{2/3}\right).}$$

Exercise 2. Compute

$$\frac{d}{dx}\int_{\sin x}^{e^{x^2+1}} t^3(\ln t)dt.$$

Solution: By the fundamental theorem of calculus, we know that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x).$$

In our problem, we see that

$$\begin{aligned} a(x) &= \sin x \\ b(x) &= e^{x^2+1} \\ f(x) &= x^3(\ln x) \end{aligned}$$

so

$$\begin{aligned} \frac{d}{dx}\int_{\sin x}^{e^{x^2+1}} t^3(\ln t)dt &= \left(e^{x^2+1}\right)^3 \ln\left(e^{x^2+1}\right)\left(e^{x^2+1}\right)' - (\sin x)^3 \ln(\sin x)(\sin x)' \\ &= \boxed{\left(e^{x^2+1}\right)^3 \ln\left(e^{x^2+1}\right)\left(e^{x^2+1}\right)(2x) - (\sin x)^3 \ln(\sin x)(\cos x).} \end{aligned}$$

Exercise 3. Find the average value of $f(x) = \sqrt{9 - x^2}$ over $[-3, 3]$ and all $-3 \leq c \leq 3$ such that $f(c) = f_{\text{avg}}$.

Solution: The formula for the average of $f(x)$ over the interval $[a, b]$ is given by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)dx.$$

So, in our problem,

$$f_{\text{avg}} = \frac{1}{3-(-3)} \int_{-3}^{3} \sqrt{9-x^2}dx = \frac{1}{6} \int_{-3}^{3} \sqrt{9-x^2}.$$

Notice that $f(x) = \sqrt{9 - x^2}$ where $x \in [-3, 3]$ is the upper half of the circle of radius $\sqrt{9} = 3$ center at the origin so $\int_{-3}^{3} f(x)dx$ is the area of this half circle. The area of the whole circle is $\pi(3)^2$ so the area of the upper half is

$$\frac{1}{2}\left(\pi(3)^2\right) = \frac{9}{2}\pi = \int_{-3}^{3} \sqrt{9-x^2}dx.$$

Thus, we have

$$f_{\text{avg}} = \frac{1}{6} \int_{-3}^{3} \sqrt{9-x^2}dx = \boxed{\frac{9\pi}{12}.}$$

To find c, we set $f(c) = f_{\text{avg}}$ and solve for c. We found f_{avg} above so we have the equation:

$$\frac{9\pi}{12} = f(c) = \sqrt{9-c^2}.$$

After squaring both sides, we get that

$$\left(\frac{9\pi}{12}\right)^2 = 9 - c^2.$$

Solving the above equation for c, we get that

$$c = \boxed{\pm\sqrt{\left(\frac{9\pi}{12}\right)^2 - 9}.}$$

Exercise 4. The following function represents the acceleration of a particle moving along the x-axis:

$$a(t) = 2t - 12 \ \text{ for } t \geq 0.$$

The starts at $x_0 = 3$ and has an initial velocity of $v_0 = 20$ units/sec.

(a) Find the velocity function $v(t)$.

(b) Find the position function $x(t)$.

(c) Find the displacement on the time interval $[0, 4]$.

(d) Find the total distance travel on the time interval $[0, 4]$.

Solution:

(a) The acceleration function is given by the derivative of the velocity so the fundamental theorem of calculus tells us that

$$v(t) = \int a(t)dt = \int 2t - 12dt = t^2 - 12t + C.$$

We are given an initial value so we can solve for C in the above equation. Namely,

$$20 = v(0) = 0^2 - 12(0) + C = C$$

which gives us our final answer

$$\boxed{v(t) = t^2 - 12t + 20.}$$

(b) The velocity function is given by the derivative of the position so the fundamental theorem of calculus tells us that

$$x(t) = \int v(t)dt = \int t^2 - 12t + 20dt = \frac{t^3}{3} - 6t^2 + 20t + C.$$

We are given an initial value so we can solve for C in the above equation. Namely,

$$3 = x(0) = \frac{0^3}{3} - 6(0)^2 + 20(0) + C = C$$

which gives us our final answer

$$\boxed{x(t) = \frac{1}{3}t^3 - 6t^2 + 20t + 3.}$$

(c) Displacement is given on the interval $[a, b]$ is given by

$$\int_a^b v(t)dt = x(b) - x(a)$$

where the above equality is given by the fundamental theorem of calculus since $x'(t) = v(t)$. We found $x(t)$ in part (b) so our total displacement on the time interval $[0, 4]$ is

$$\int_0^4 v(t)dt = x(4) - x(0) = \frac{1}{3}4^3 - 6(4)^2 + 20(4) + 3 - 3 = \boxed{\frac{1}{3}4^3 - 6(4)^2 + 20(4).}$$

(d) Total distance traveled on $[a, b]$ is given by

$$\int_a^b |v(t)| dt$$

so we need to evaluate

$$\int_0^4 \left| t^2 - 12t + 20 \right| dt.$$

To deal with the absolute value in the above integral, we need to split $[0, 4]$ into parts where $t^2 - 12t + 20$ is either entirely negative or entirely positive. To do so, we find the zeros of $t^2 - 12t + 20$. The quadratic formula tells us that the zeros are given by

$$\frac{12 \pm \sqrt{12^2 - 4(20)}}{2} = \frac{12 \pm 4\sqrt{3^2 - 5}}{2} = \frac{12 \pm 8}{2} = 6 \pm 4.$$

Of those two zeros, only 2 is in the interval $[0, 4]$ so we need to split our integral at 2. Since $v(0) = 20 > 0$ and $v(3) = 9 - 36 + 20 = -7 < 0$, we see that $|v(t)| = v(t)$ when $0 \leq t \leq 2$ and $|v(t)| = -v(t)$ when $2 \leq t \leq 4$. Thus, we have

$$\begin{aligned}
\int_0^4 |v(t)| dt &= \int_0^2 v(t) dt - \int_2^4 v(t) dt \\
&= d(2) - d(0) - (d(4) - d(2)) \qquad \text{[by FTC]} \\
&= 2d(2) - d(4) - d(0) \\
&= \boxed{2\left(\frac{1}{3}2^3 - 6(2)^2 + 20(2) + 3\right) - \left(\frac{1}{3}4^3 - 6(4)^2 + 20(4) + 3\right) - 3.}
\end{aligned}$$

Exercise 5. Evaluate
$$\int x\sin(\ln x)dx.$$

Solution: Notice that $\ln x$ is our "inside" function here so we begin by making the substitution:
$$t = \ln x$$
$$dt = \frac{1}{x}dx$$
Solving for x in terms of t, we see that $x = e^t$. Solving for dx in terms of t, we see that $dx = xdt = e^t dt$ so our integral, which we'll call I, becomes
$$I = \int e^t(\sin t)\left(e^t dt\right) = \int e^{2t}\sin t dt.$$
Now, we have an integral with no "inside" functions and whose integrand is the product of two function so we should apply integration by parts. Recalling LIPET, we choose u to be e^{2t} so dv is the rest of our integrand which gives us:
$$\begin{aligned} u &= e^{2t} & v &= -\cos t \\ du &= 2e^{2t}dt & dv &= \sin t dt \end{aligned}$$
Applying the integration by parts formula, we have
$$I = -e^{2t}\cos t - \int(-\cos t)(2e^{2t})dt = -e^{2t}\cos t + 2\int e^{2t}\cos t dt \tag{1}$$
We once again have an integral with no "inside" functions and whose integrand is the product of two function so we should apply integration by parts. Once again recalling LIPET, we have:
$$\begin{aligned} u &= e^{2t} & v &= \sin t \\ du &= 2e^{2t}dt & dv &= \cos t dt \end{aligned}$$
Applying the integration by parts formula, we have
$$\int e^{2t}\cos t dt = e^{2t}\sin t - \int 2e^{2t}\sin t dt = e^{2t}\sin t - 2\int e^{2t}\sin t dt = e^{2t}\sin t - 2I + C.$$
Plugging this into equation (1), we see that
$$I = -e^{2t}\cos t + 2\int e^{2t}\cos t dt = -e^{2t}\cos t + \left(e^{2t}\sin t - 2I + C\right).$$
Solving for I, we find that
$$I = \frac{1}{3}\left(e^{2t}\sin t - e^{2t}\cos t\right) + C.$$
Undoing our substitution $t = \ln x$, we get the final answer
$$I = \boxed{\frac{1}{3}\left(e^{2\ln t}\sin(\ln t) - e^{2\ln t}\cos(\ln t)\right) + C.}$$

Exercise 6. Evaluate

$$\int (x^2 + x + 1) \ln x dx.$$

Solution: Anytime we want to integrate the product of a polynomial and a transcendental function, integration by parts is the best place to start. Recalling LIPET, we set:

$$\begin{array}{cc} u = \ln x & v = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \\ du = \frac{1}{x}dx & dv = (x^2 + x + 1)dx \end{array}$$

Applying the integration by parts formula, we see that

$$\begin{aligned} \int (x^2 + x + 1) \ln x dx &= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right) \ln x - \int \frac{1}{x} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right) dx \\ &= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right) \ln x - \int \frac{1}{3}x^2 + \frac{1}{2}x + 1 dx \\ &= \boxed{\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right) \ln x - \left(\frac{1}{3^2}x^3 + \frac{1}{2^2}x^2 + x\right) + C.} \end{aligned}$$

Exercise 7. Evaluate

$$\int \frac{1}{(e^t - e^{-t})^2} dt.$$

Solution: We begin by simplifying the integrand:

$$\begin{aligned}
\frac{1}{(e^t - e^{-t})^2} &= \frac{1}{\left(e^t - \frac{1}{e^t}\right)^2} \\
&= \frac{1}{\left(\frac{e^{2t}-1}{e^t}\right)^2} \\
&= \frac{e^{2t}}{(e^{2t} - 1)^2}.
\end{aligned}$$

Now, we see that $e^{2t} - 1$ is an "inside" function so consider the substitution

$$\begin{aligned}
u &= e^{2t} - 1 \\
du &= 2e^{2t} dt.
\end{aligned}$$

This gives us

$$\begin{aligned}
\int \frac{1}{(e^t - e^{-t})} dt = \int \frac{e^{2t}}{(e^{2t} - 1)^2} dt = \frac{1}{2} \int \frac{1}{u^2} du \\
&= \frac{1}{2} \left(-\frac{1}{-2+1} u^{-2+1} \right) + C \\
&= -\frac{1}{2u} + C \\
&= \boxed{-\frac{1}{2(e^{2t} - 1)} + C}
\end{aligned}$$

where the final equality follows from our substitution: $u = e^{2t} - 1$.

Exercise 8. Evaluate

$$\int (x^3 - x + 1)e^{2x}dx.$$

Solution: Whenever we want to integrate the product of a polynomial and a transcendental function, we first attempt integration by parts. Recalling LIPET, we take u to be the polynomial. Moreover, as taking the derivative of a polynomial eventually results in 0, we will employ the tabular method. We get:

u	dv
$x^3 - x + 1$	e^{2x}
$3x^2 - 1$	$\frac{1}{2}e^{2x}$
$6x$	$\frac{1}{2^2}e^{2x}$
6	$\frac{1}{2^3}e^{2x}$
0	$\frac{1}{2^4}e^{2x}$

Remembering to alternate signs, we get that

$$\int (x^3 - x + 1)e^{2x}dx = \boxed{\frac{1}{2}(x^3 - x + 1)e^{2x} - \frac{1}{2^2}(3x^2 - 1)e^{2x} + \frac{1}{2^3}(6x)e^{2x} - \frac{6}{2^4}e^{2x} + C.}$$

Exercise 9. Find the volume of the shape obtained by revolving the area between

$$f(x) = \sin x \ \text{ and } \ g(x) = \cos x$$

around the x-axis for $0 \leq x \leq \frac{\pi}{2}$.

Solution: Referencing an unit circle, we see that $\sin x = \cos x$ only when $x = \frac{\pi}{4}$ for $0 \leq x \leq \frac{\pi}{2}$. Since $\cos(0) = 1$ and $\sin(0) = 0$, $\cos x$ is the "top" function while $\sin x$ is the "bottom" function for $0 \leq x \leq \pi/4$ and they switch for $\pi/4 \leq x \leq \pi/2$. Thus, the volume of this shape is given by summing the volume of revolving the area between $f(x)$ and $g(x)$ for x in $[0, \pi/4]$ and $[\pi/4, \pi/2]$ separately. Hence, we have

$$\begin{aligned}
\text{Volume} &= \int_0^{\pi/4} \pi \left(\cos x - \sin x\right)^2 dx + \int_{\pi/4}^{\pi/2} \pi \left(\sin x - \cos x\right)^2 dx \\
&= \pi \left[\int_0^{\pi/4} (\cos x - \sin x)^2 dx + \int_{\pi/4}^{\pi/2} (\cos x - \sin x)^2 dx \right] \\
&= \pi \int_0^{\pi/2} (\cos x - \sin x)^2 dx \\
&= \pi \int_0^{\pi/2} \big((\cos x)^2 + (\sin x)^2 - 2(\sin x)(\cos x)\big)\, dx \\
&= \pi \int_0^{\pi/2} 1 - 2(\sin x)(\cos x) dx \\
&= \frac{\pi^2}{2} - 2\pi \int_0^{\pi/2} (\sin x)(\cos x) dx.
\end{aligned}$$

Finally, we make the substitution:

$$\begin{aligned}
u &= \sin x \\
du &= \cos x dx
\end{aligned}$$

so our bounds become $\sin(0) = 0$ and $\sin(\pi/2) = 1$. Thus,

$$\begin{aligned}
\text{Volume} &= \frac{\pi^2}{2} - 2\pi \int_0^{\pi/2} (\sin x)(\cos x) dx \\
&= \frac{\pi^2}{2} - 2\pi \int_0^1 u du \\
&= \frac{\pi^2}{2} - 2\pi \left(\frac{1}{2} u^2 \Big|_0^1 \right) \\
&= \boxed{\frac{\pi^2}{2} - \pi.}
\end{aligned}$$

Exercise 10. Find the area between

$$f(x) = x^3 \text{ and } g(x) = x$$

for $0 \leq x \leq 10$.

Solution: Graphing both $f(x)$ and $g(x)$, we see the $g(x) \leq f(x)$ for $0 \leq x \leq 1$ while $f(x) \leq g(x)$ for $1 \leq x \leq 10$. So, the area between f and g over $[0, 10]$ is given by

$$\begin{aligned}
\int_0^{10} |f(x) - g(x)| dx &= \int_0^1 x - x^3 dx + \int_1^{10} x^3 - x dx \\
&= \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_0^1 + \frac{1}{4}x^4 - \frac{1}{2}x^2 \Big|_1^{10} \\
&= \boxed{\frac{1}{2} - \frac{1}{4} + \left(\frac{1}{4}10^4 - \frac{1}{2}10^2 - \frac{1}{4} + \frac{1}{2}\right)}.
\end{aligned}$$

Exercise 11. Evaluate
$$\int \frac{x}{x^2 - x - 2} dx.$$

Solution: Notice that the integrand is a rational function and there's no obvious substitution to simplify the denominator into a linear polynomial, so we employ parital fractions. Note that $x^2 - x - 2$ factors as $(x-2)(x+1)$ so

$$\frac{x}{x^2 - x - 2} = \frac{A}{x-2} + \frac{B}{x+1} \Longrightarrow x = A(x+1) + B(x-2)$$

where the right equation is obtained by multiplying both sides of the left equation by $x^2 - x - 2 = (x-2)(x+1)$. Evaluating the equation on the left at $x = -1$ and $x = 2$ gives us:

$$-1 = A(-1+1) + B(-1-3) = -4B \Longrightarrow B = \frac{1}{4}$$
$$2 = A(2+1) + B(2-2) = 3A \Longrightarrow A = \frac{2}{3}$$

Thus, our integral becomes

$$\int \frac{x}{x^2 - x - 2} dx = \int \frac{2/3}{x-2} dx + \int \frac{1/4}{x+1} dx = \boxed{\frac{2}{3} \ln |x-2| + \frac{1}{4} \ln |x+1| + C.}$$

Exercise 12. Evaluate
$$\int \frac{1}{x((\ln x)^2 - 3(\ln x) + 2)} dx.$$

Solution: We see that $\ln x$ is an "inside" function of our integrand so we make the substitution

$$u = \ln x$$
$$du = \frac{1}{x} dx$$

So, our integral becomes

$$\int \frac{1/x}{(\ln x)^2 - 3(\ln x) + 2} dx = \int \frac{1}{u^2 - 3u + 2} du.$$

Now, we have a rational function and no obvious substitution so we employ partial fractions. Note that $u^2 - 3u + 2 = (u-1)(u-2)$ so we have

$$\frac{1}{u^2 - 3u + 2} = \frac{A}{u-1} + \frac{B}{u-2} \Longrightarrow 1 = A(u-2) + B(u-1).$$

Evaluating the equation on the left at $u = 2$ and $u = 1$, we get

$$1 = A(2-2) + B(2-1) = B \Longrightarrow B = 1$$
$$1 = A(1-2) + B(1-1) = -A \Longrightarrow A = -1$$

Thus, our integral becomes

$$\begin{aligned} \int \frac{1}{u^2 - 3u + 2} du &= \int \frac{-1}{u-1} du + \int \frac{1}{u-2} du \\ &= -\ln|u-1| + \ln|u-2| + C \\ &= \boxed{-\ln|\ln x - 1| + \ln|\ln x - 2| + C} \end{aligned}$$

where the final equality comes from our substitution: $u = \ln x$.

Exercise 13. Determine whether the following improper integrals converge or diverge:

(a)
$$\int_1^\infty \frac{1}{x^{50}+2x-1}dx$$

(b)
$$\int_1^\infty \frac{(\ln x)^{123}}{x}dx$$

(c)
$$\int_1^3 \frac{1}{(x-2)^{1/5}}dx$$

Solution:

(a) Note that
$$\frac{1}{x^{50}+2x-1} \approx \frac{1}{x^{50}}$$
when x is large. Integrating $1/x^{50}$ over $[1,\infty)$ results in a finite value so we conjecture that our original integral does as well. To prove this, we use the comparison theorem. Since $x \geq 1$, we know that
$$\frac{1}{x^{50}+2x-1} < \frac{1}{x^{50}}$$
as $2x-1 \geq 2(1)-1 = 1 > 0$. By the comparison theorem, we have
$$\int_1^\infty \frac{1}{x^{50}+2x-1}dx \leq \int_1^\infty \frac{1}{x^{50}}dx = \lim_{t\to\infty} \frac{-1}{49}x^{-49}\bigg|_1^t = \lim_{t\to\infty} -\frac{1}{49t^{49}} - \left(-\frac{1}{49}\right) = \frac{1}{49} < \infty.$$
Hence, the integral $\boxed{\text{converges.}}$

(b) Note that $1 \leq (\ln x)^{123}$ for $x \geq e$ as $\ln x$ is an increasing function. So, we see that
$$\frac{1}{x} \leq \frac{(\ln x)^{123}}{x}$$
for $x \geq e$. By the comparison theorem, we see that
$$\int_e^\infty \frac{(\ln x)^{123}}{x}dx \geq \int_e^\infty \frac{1}{x}dx = \infty.$$
Since
$$\int_1^\infty \frac{(\ln x)^{123}}{x}dx = \int_1^e \frac{(\ln x)^{123}}{x}dx + \int_e^\infty \frac{(\ln x)^{123}}{x}dx,$$
we conclude our integral $\boxed{\text{diverges}}$ as
$$\int_1^e \frac{(\ln x)^{123}}{x}dx < \infty \quad \text{and} \quad \int_e^\infty \frac{(\ln x)^{123}}{x}dx = \infty.$$

(c) We begin by noting that

$$\int \frac{1}{(x-2)^{1/5}}dx = \frac{1}{-1/5+1}(x-2)^{-1/5+1} + C = \frac{5}{4}(x-2)^{4/5} + C.$$

Therefore, we see that

$$\begin{aligned}\int_1^3 \frac{1}{(x-2)^{1/5}}dx &= \lim_{t\to 2^-} \frac{5(x-2)^{4/5}}{4}\Bigg|_1^t + \lim_{t\to 2^+} \frac{5(x-2)^{4/5}}{4}\Bigg|_t^3 \\ &= \lim_{t\to 2^-} \frac{5(t-2)^{4/5}}{4} - \frac{5}{4} + \frac{5}{4} - \lim_{t\to 2^+} \frac{5(t-2)^{4/5}}{4} \\ &= 0 < \infty\end{aligned}$$

so our integral $\boxed{\text{converges.}}$

Exercise 14. Solve for y with $y(0) = 1$:

(a)
$$\frac{dy}{dx} = (1 + 4x^3)\frac{y}{\ln y}$$

(b)
$$\frac{dy}{dx} = \frac{yx}{1 + x^2}$$

Solution:

(a) "Multiplying" both sides by $\frac{\ln y}{y} dx$, we get
$$\frac{\ln y}{y} dy = 1 + 4x^3 dx.$$
Integrating both sides, we get
$$\int \frac{\ln y}{y} dy = \frac{(\ln y)^2}{2} = \int 1 + 4x^3 dx = x^4 + x + C.$$
Solving for y, we find that
$$y = e^{\pm\sqrt{x^4+x+C}}.$$
Using the fact that $y(0) = 1$, we see that
$$1 = y(0) = e^{\pm\sqrt{0^4+0+C}} = e^{\pm\sqrt{C}} \Longrightarrow \pm\sqrt{C} = 0 \Longrightarrow C = 0.$$
Giving us the final answer,
$$\boxed{y = e^{\pm\sqrt{x^4+x}}.}$$

(b) "Multiplying" both sides by $\frac{dx}{y}$, we get
$$\frac{1}{y} dy = \frac{x}{1 + x^2} dx.$$
Integrating both sides, we get
$$\int \frac{1}{y} dy = \ln y = \int \frac{x}{1 + x^2} dx = \frac{1}{2} \ln |1 + x^2| + C.$$
Solving for y, we find that
$$y = Ce^{\frac{1}{2}\ln|1+x^2|} = C|1 + x^2|^{1/2} = C(1 + x^2)^{1/2}$$
as $1 + x^2 \geq 0$ for all x. Finally, using the fact that $y(0) = 1$, we solve for C:
$$1 = y(0) = C(1 + 0^2)^{1/2} = C.$$
Thus, our final answer is
$$\boxed{y = \sqrt{1 + x^2}.}$$

Exercise 15. Given a tank with 250L and 4kg of salt mixed in, suppose that 4L is added to the tank every minute which contains $\frac{1}{4}$kg/L of salt. Moreover, 4L is drained out of the tank every minute so the amount of liquid remains constant. Find the concentration function for salt in the tank as a function of time.

Solution: Let $u(t)$ denote the amount of salt in the tank at time t then

$$\frac{du}{dt} = (\text{in flow/min}) - (\text{out flow/min}) = \frac{1}{4}(4) - \frac{u}{250}(4) = 1 - \frac{2u}{125} = \frac{125-2u}{125}.$$

"Multiplying" both sides by $\frac{dt}{125-2u}$, we get that

$$\frac{1}{125-2u}du = \frac{1}{125}dt.$$

Integrating both sides, get that

$$\int \frac{1}{125-2u}du = -\frac{1}{2}\ln|125-2u| = \frac{t}{125} + C.$$

Solving for u, we see that

$$|125-2u|^{-1/2} = Ce^{t/125} \Longrightarrow 125 - 2u = Ce^{-2t/125} \Longrightarrow \frac{1}{2}\left(125 - Ce^{-2t/125}\right) = u.$$

Since the tank has 4kg of salt initially, we see that

$$4 = u(0) = \frac{1}{2}\left(125 - Ce^{-2(0)/125}\right) = \frac{1}{2}(125 - C) \Longrightarrow C = 125 - 8 = 117.$$

Thus, we find that

$$u = \frac{1}{2}\left(125 - 117e^{-2t/125}\right).$$

Finally, the salt concentration is given by u over the total amount of liquid in the tank:

$$\frac{u}{250} = \boxed{\frac{1}{500}\left(125 - 117e^{-2t/125}\right).}$$