Exercise 1. Evaluate the following improper integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx$

(b) $\displaystyle\int_{3}^{4} \frac{5}{(x-4)^2}dx$

(c) $\displaystyle\int_{0}^{1} \frac{\ln x}{x}dx$

Solutions:

(a) First, we need to compute

$$\int \frac{e^x}{1+e^{2x}}dx.$$

To do so, we use the substitution:

$$u = e^x \implies du = e^x dx \implies dx = \frac{du}{e^x} = \frac{du}{u}.$$

Hence, we see that

$$\begin{aligned}
\int \frac{e^x}{1+e^{2x}}dx &= \int \frac{u}{1+u^2}\frac{du}{u} && \text{[using our substitution]}\\
&= \int \frac{1}{1+u^2}du\\
&= \arctan(u) + C && \text{[known integral formula]}\\
&= \arctan(e^x) + C. && \text{[as } u = e^x\text{]}
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx &= \lim_{t\to\infty} \arctan\left(e^t\right) - \lim_{x\to-\infty} \arctan(e^x)\\
&= \lim_{t\to\infty} \arctan(t) - \lim_{x\to-\infty} \arctan(e^x) && \text{[as } e^t \to \infty \text{ as } t \to \infty\text{]}\\
&= \frac{\pi}{2} - \lim_{x\to-\infty} \arctan(e^x) && \text{[known limit]}\\
&= \frac{\pi}{2} - \arctan\left(\lim_{x\to-\infty} e^x\right) && \text{[as arctan is continuous]}\\
&= \frac{\pi}{2} - \arctan(0) && \text{[known limit]}\\
&= \boxed{\frac{\pi}{2}.}
\end{aligned}$$

(b) Note that

$$\left.\frac{5}{(x-4)^2}\right|_{x=4}$$

is undefined as $(x-4)^2|_{x=4} = 0$. So, this integral is, in fact, improper and

$$\int_3^4 \frac{5}{(x-4)^2}dx = \lim_{t\to4^-}\int_3^t \frac{5}{(x-4)^2}dx.$$

Now, we want to evaluate the indefinite integral: $\int \frac{5}{(x-4)^2} dx$. Using the substitution $u = x - 4$, we get that $du = dx$ so

$$\begin{aligned}
\int \frac{5}{(x-4)^2} dx &= \int \frac{5}{u^2} du && \text{[using our substitution]} \\
&= -5u^{-1} + C && \text{[by the anti-power rule]} \\
&= \frac{-5}{x-4} + C. && \text{[as } u = x - 4\text{]}
\end{aligned}$$

Now, we see that

$$\begin{aligned}
\int_3^4 \frac{5}{(x-4)^2} dx &= \lim_{t \to 4^-} \int_3^t \frac{5}{(x-4)^2} dx \\
&= \lim_{t \to 4^-} \frac{-5}{t-4} - \frac{-5}{3-4} && \text{[by FTC]} \\
&= 5 + 5 \lim_{t \to 4^-} \frac{1}{4-t} \\
&= \boxed{\infty}
\end{aligned}$$

so the integral diverges.

(c) Note that $\frac{\ln x}{x}$ is undefined at $x = 0$ so our integral is improper and

$$\int_0^1 \frac{\ln x}{x} dx = \lim_{t \to 0^+} \int_t^1 \frac{\ln x}{x} dx.$$

We'll use the substitution:

$$u = \ln x \Longrightarrow du = \frac{dx}{x} \Longrightarrow dx = xdu$$

to evaluate the indefinite integral. Note that

$$\begin{aligned}
\int \frac{\ln x}{x} dx &= \int \frac{u}{x}(xdu) && \text{[by our substitution]} \\
&= \int udu \\
&= \frac{u^2}{2} + C && \text{[by anti-power rule]} \\
&= \frac{(\ln x)^2}{2} + C. && \text{[as } u = \ln x\text{]}
\end{aligned}$$

Hence, we have

$$\int_0^1 \frac{\ln x}{x} dx = \lim_{t \to 0^+} \int_t^1 \frac{\ln x}{x} dx = \frac{\ln 1}{1} - \lim_{t \to 0^+} \frac{(\ln x)^2}{2} = \boxed{-\infty}$$

so our integral diverges.

Exercise 2. Find $y(x)$ satisfying the given differential equation (solving for C when initial conditions are given):

(a) $(1+x)\dfrac{dy}{dx} = (2+x)(y-1)$ with $y(0) = 2$

(b) $\dfrac{dy}{dx} = y\cos(3x+2)$

(c) $\dfrac{dy}{dx} = e^{y-x}$ with $y(0) = 0$

Solutions:

(a) First, we move all x-terms to the right and all y-terms to the left (with dy/dx):

$$(1+x)\frac{dy}{dx} = (2+x)(y-1) \implies \frac{1}{y-1}\frac{dy}{dx} = \frac{2+x}{1+x}.$$

Hence, we see that

$$\begin{aligned}\int \frac{1}{y-1}\frac{dy}{dx}dx = \int \frac{1}{y-1}dy &= \ln|y-1| \\ &= \int \frac{2+x}{1+x}dx \\ &= \int \frac{1+x}{1+x} + \frac{1}{1+x}dx \\ &= \int 1 + \frac{1}{1+x}dx \\ &= x + \ln|1+x| + C.\end{aligned}$$

Now, solving for y in the above:

$$|y-1| = e^{\ln|y-1|} = e^{x+\ln|1+x|+C} = C_0 e^x e^{\ln|1+x|} = C_0 e^x|1+x| \implies y = 1 \pm C_0 e^x|1+x|.$$

where $C_0 = e^{C_1} > 0$. Using our initial conditions, we find

$$2 = y(0) = 1 \pm C_0 e^0|1+0| = 1 \pm C_1 \implies \pm C_1 = 1.$$

Thus, we conclude that

$$\boxed{y = 1 + e^x|1+x|.}$$

(b) First, we move all our x-terms to the left and all y-terms to the left:

$$\frac{dy}{dx} = y\cos(3x+2) \implies \frac{1}{y}\frac{dy}{dx} = \cos(3x+2).$$

Hence, we see that

$$\begin{aligned}\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{y}dy &= \ln|y| \\ &= \int \cos(3x+2)dx \\ &= \frac{1}{3}\sin(3x+2) + C.\end{aligned}$$

Try using u-substitution to evaluate $\int \cos(3x+2)dx$ yourself! Now, solving for y, we see:

$$|y| = e^{\ln|y|} = e^{\frac{1}{3}\sin(3x+2)+C} = C_0 e^{\frac{1}{3}\sin(3x+2)} \Longrightarrow y = \pm C_0 e^{\frac{1}{3}\sin(3x+2)}$$

where $C_0 = e^C > 0$. As C_0 is an arbitrary positive number, $\pm C_0$ is just some constant real number so

$$\boxed{y = Ce^{\frac{1}{3}\sin(3x+2)}.}$$

(c) First, we move all our x-terms to the left and all y-terms to the left:

$$\frac{dy}{dx} = e^{y-x} = e^y e^{-x} \Longrightarrow e^{-y}\frac{dy}{dx} = e^{-x}.$$

Hence, we see that

$$\begin{aligned} \int e^{-y}\frac{dy}{dx}dx = \int e^{-y}dy &= -e^{-y} \\ &= \int e^{-x}dx \\ &= -e^{-x} + C. \end{aligned}$$

Now, solving for y, we get:

$$-y = \ln e^{-y} = \ln\left(e^{-x} + C\right) \Longrightarrow y = -\ln\left(e^{-x} + C\right).$$

Using our initial conditions, we see that

$$0 = y(0) = -\ln(e^{-0} + C) = -\ln(1 + C) \Longrightarrow 1 + C = e^{\ln(1+C)} = e^0 = 1 \Longrightarrow C = 0.$$

Thus, we conclude that

$$\boxed{y = -\ln e^{-x} = x.}$$