Exercise 1. Find the volume of the solid obtained by rotating the area between $y = 3\arcsin x^2$, $y = 0$, and $x = 1$ around the <u>y-axis</u>.

With these kinds of problems, it's always best to have a picture to reference. Try sketching it out!

Solution: Since we are rotating about the y-axis, we need our functions in terms of y, rather than x:

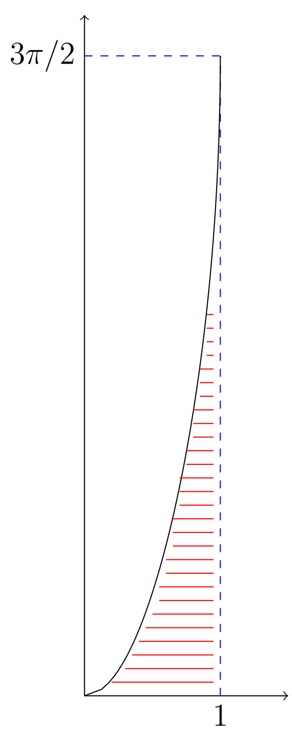
$$y = 3\arcsin x^2 \implies \frac{y}{3} = \arcsin x^2 \implies \sin\left(\frac{y}{3}\right) = x^2 \implies \sqrt{\sin\left(\frac{y}{3}\right)} = x.$$

Now, our bounds will be in terms of y. We are given one of them, i.e. $y = 0$, we need to find the other, that is, the point where $x = \sqrt{\sin(y/3)}$ and $x = 1$ intersect:

$$1 = \sqrt{\sin(y/3)} \implies 1 = \sin(y/3) \implies \frac{y}{3} = \frac{\pi}{2} \implies y = \frac{3\pi}{2}.$$

Remark: Since $\arcsin x$ *has a range of* $[-\pi/2, \pi/2]$, *when we see that* $1 = \sin(y/3)$ *we know that a solution must lie in the interval* $[-\pi/2, \pi/2]$. *Hence, we know* $y/3 = \pi/2$ *rather than another solution (i.e.* $y/3 = 5\pi/2$*).* Finally, as $\sin(y/3) \le 1$ for all y, we see that the volume of this region is given by:

$$\begin{aligned}
\int_0^{3\pi/2} \pi\left(1^2 - \left(\sqrt{\sin(y/3)}\right)^2\right) dy &= \pi \int_0^{3\pi/2} (1 - \sin(y/3))\, dy \\
&= \pi\left(x + 3\cos(y/3)\right)\Big|_0^{3\pi/3} \qquad \text{[try } u = y/3 \text{ to see this]} \\
&= \pi\left(\frac{3\pi}{2} + 3(0) - (0 + 3(1))\right) \\
&= \boxed{\pi\left(\frac{3\pi}{2} - 3\right).}
\end{aligned}$$



Exercise 2. Find the volume of the solid obtained by rotating the area between $y = x^2 - 2$ and $y = x - 2$ around the <u>x-axis</u>.

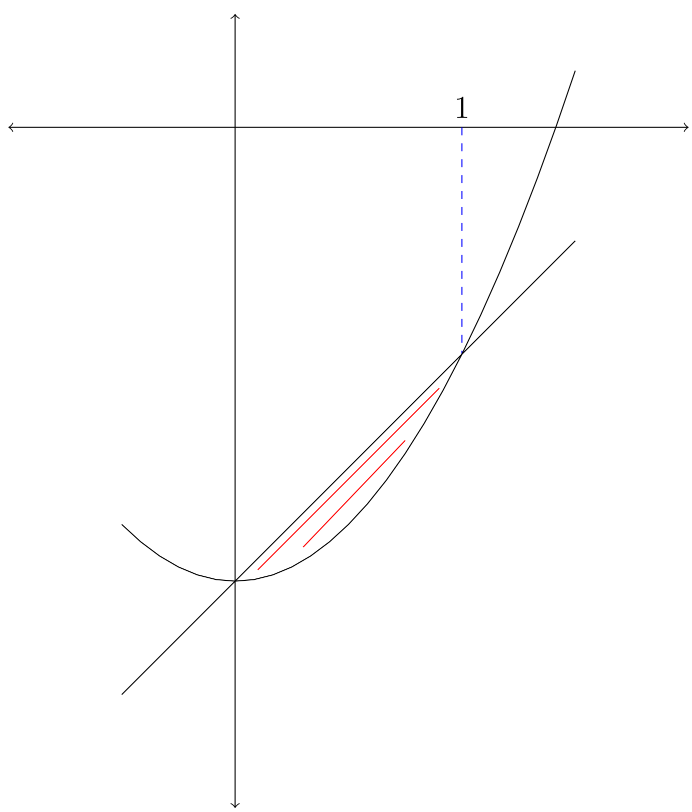
With these kinds of problems, it's always best to have a picture to reference. Try sketching it out!

Solution: We are not given bounds for our integration, to find these we need the point(s) where $y = x^2 - 2$ and $y = x - 2$ intersect:

$$x^2 - 2 = x - 2 \Longrightarrow x^2 - x = x(x-1) = 0 \Longrightarrow x = 0, 1.$$

Hence, our bounds are $x = 0$ and $x = 1$. Moreover, we see that $(1/2)^2 - 2 = -7/4 < -3/2 = (1/2) - 2$ so $x^2 - 2 \le x - 2$ on $[0, 1]$. Thus, our volume is given by

$$\begin{aligned}
\int_0^1 \pi \left((x-2)^2 - \left(x^2-2\right)^2\right) dx &= \pi \int_0^1 \left(x^2 - 4x + 4 - x^4 + 4x^2 - 4\right) dx \\
&= \pi \int_0^1 5x^2 - 4x - x^4 dx \\
&= \pi \left(\frac{5x^3}{3} - 2x^2 - \frac{x^5}{5}\right) \Big|_0^1 \\
&= \boxed{\pi \left(\frac{5}{3} - 2 - \frac{1}{5}\right).}
\end{aligned}$$



Exercise 3. Evaluate the following integrals:

(a) $\int x^3 \ln x dx$

(b) $\int xe^{-x} dx$

(c) $\int e^x \cos x dx$

Solutions:

(a) Following *LIATE*, we should take $u = \ln x$ and $dv = x^3 dx$ which gives us:

$$\boxed{\begin{array}{cc} u = \ln x & v = x^4/4 \\ du = dx/x & dv = x^3 dx \end{array}}$$

Now, our integral becomes:

$$\begin{aligned} \int x^3 \ln x dx &= \ln x \left(\frac{x^4}{4}\right) - \int \frac{1}{x}\left(\frac{x^4}{4}\right) dx \\ &= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} dx \\ &= \boxed{\frac{x^4 \ln x}{4} - \frac{x^4}{16} + C.} \end{aligned}$$

(b) Following *LIATE*, we take $u = x$ and $dv = e^{-x} dx$ which gives us:

$$\boxed{\begin{array}{cc} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x} dx \end{array}}$$

So, our integral becomes

$$\int xe^{-x} dx = -xe^{-x} - \int \left(-e^{-x}\right) dx = -xe^{-x} + \int e^{-x} dx = \boxed{-xe^{-x} - e^{-x} + C.}$$

(c) Following *LIATE*, we take $u = \cos x$ and $dv = e^x dx$ which gives:

$$\boxed{\begin{array}{cc} u = \cos x & v = e^x \\ du = -\sin x dx & dv = e^x dx \end{array}}$$

So, our integral becomes

$$\int e^x \cos x dx = e^x \cos x - \int e^x(-\sin x) dx = e^x \cos x + \int e^x \sin x dx.$$

As $\int e^x \sin x dx$ takes a similar form to our original integral, let's do the same thing again (following *LIATE*):

$$\boxed{\begin{array}{cc} u = \sin x & v = e^x \\ du = \cos x dx & dv = e^x dx \end{array}}$$

Therefore, we have

$$\int e^x \sin x = e^x \sin x - \int e^x \cos x dx$$

so, after plugging back, we have

$$\int e^x \cos dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx.$$

Solving for our goal integral, we find

$$\int e^x \cos x dx = \boxed{\frac{1}{2}\left(e^x \cos x + e^x \sin x\right) + C.}$$