Recall that:

$$\begin{aligned} \int \frac{1}{1+t^2}dt &= \arctan t + C && (1)\\ \int \frac{1}{\sqrt{1-t^2}}dt &= \arcsin t + C && (2)\\ \int \frac{1}{|t|\sqrt{1-t^2}}dt &= \operatorname{arcsec} t + C && (3) \end{aligned}$$

Exercise 1. Evaluate:

(a) $\displaystyle\int \frac{1}{t\sqrt{4-(\ln t)^2}}dt$

(b) $\displaystyle\int (\cos x)^3 dx$

(c) $\displaystyle\int_0^{\pi} \frac{\sin t}{1+9(\cos t)^2}dt$

Solutions:

(a) First, we want the 4 in the integrand to be 1 (we'll see why in a second) so we write

$$\frac{1}{t\sqrt{4-(\ln t)^2}} = \frac{1}{t\sqrt{4\left(1-\left(\frac{\ln t}{2}\right)^2\right)}} = \frac{1}{2}\left(\frac{1}{t\sqrt{1-\left(\frac{\ln t}{2}\right)^2}}\right).$$

Therefore we need only integrate

$$\frac{1}{t\sqrt{1-\left(\frac{\ln t}{2}\right)^2}}$$

then multiply the result by 1/2. We see that $\frac{\ln t}{2}$ is "inside" as we're squaring it so let's take that as our u then

$$du = \frac{1}{2t}dt \Longrightarrow dt = 2tdu.$$

Then, our integral becomes

$$\begin{aligned} \int \frac{1}{t\sqrt{1-u^2}}(2t)du &= 2\int \frac{1}{\sqrt{1-u^2}}du \\ &= 2\arcsin(u) + C && \text{[by (2)]}\\ &= 2\arcsin\left(\frac{\ln t}{2}\right) + C. && [u = \ln t/2] \end{aligned}$$

Therefore, we conclude that

$$\int \frac{1}{t\sqrt{4-(\ln t)^2}}dt = \frac{1}{2}\left(2\arcsin\left(\frac{\ln t}{2}\right) + C\right) = \boxed{\arcsin\left(\frac{\ln t}{2}\right) + C.}$$

(b) We'd like to make some sort of substitution here but they obvious choice ($\cos x$) doesn't have a du which results in nice cancellation. So, we use the following identity to get a $\sin x$ term:

$$1 = (\cos x)^2 + (\sin x)^2 \Longrightarrow (\cos x)^2 = 1 - (\sin x)^2.$$

Therefore, our integral becomes

$$\int (\cos x)^3 dx = \int (\cos x)(1-(\sin x)^2)dx.$$

Now, if we take $u = \sin x$ then

$$du = \cos x dx \Longrightarrow \frac{du}{\cos x} = dx.$$

Hence, we have

$$\begin{aligned}
\int (\cos x)^3 dx &= \int (\cos x)(1-(\sin x)^2)dx \\
&= \int (\cos x)(1-u^2)\frac{du}{\cos x} && \text{[using our substitution]} \\
&= \int 1-u^2 du \\
&= u - \frac{u^3}{3} + C && \text{[using the antipower rule]} \\
&= \boxed{\sin x - \frac{(\sin x)^3}{3} + C.} && [u = \sin x]
\end{aligned}$$

(c) Note that the derivative of $\cos t$ is $-\sin t$ which appear in our integrand so our substitution should involve $\cos t$. But, we can include a factor of 3 to reduce a little further without really changing the resulting du. So, let's take $u = 3\cos t$ then

$$du = -3\sin t dt \Longrightarrow dt = \frac{du}{-3\sin t}.$$

And, our bounds become $u(0) = 3$ and $u(\pi) = -3$ Hence, our integral becomes

$$\begin{aligned}
\int_0^\pi \frac{\sin t}{1+(3\cos t)^2}dt &= \int_3^{-3} \frac{\sin t}{1+u^2}\left(\frac{du}{-3\sin t}\right) && \text{[by our substitution]} \\
&= -\frac{1}{3}\int_3^{-3} \frac{1}{1+u^2}du \\
&= -\frac{1}{3}\left(\arctan u\right)\Big|_3^{-3} && \text{[by (1) and FTC]} \\
&= \boxed{\frac{1}{3}\arctan(3) - \frac{1}{3}\arctan(-3)}
\end{aligned}$$

Exercise 2. Find the area between the given curves on the given interval (if given):

(a) $y = x^2$ and $y = 18x - x^2$

(b) $y = \cos x$ and $y = (\cos x)^2$ on $[0, \pi]$

(c) $y = e^x$ and $y = e^{-x}$ on $[-1, 1]$

Solutions:

(a) As we're not given an explicit interval, we see to find the x-values where the curves intersect. That is, we want solutions to

$$x^2 = 18x - x^2 \Longrightarrow 2x^2 - 18x = 0 \Longrightarrow x^2 - 9x = x(x-9) = 0$$

so $x = 0, 9$. So, the bounds of our integration are 0 and 9. Now, we see that $18(1) - (1)^2 = 17 > 1^2$ so $18x - x^2 \geq x^2$ for $x \in [0, 9]$. Thus, our area is given by

$$\int_0^9 (18x - x^2) - x^2 dx = \int_0^9 18x - 2x^2 dx = 9x^2 - \frac{2x^3}{3}\Big|_0^9 = \boxed{9^3 - 2(3)(9^2)}$$

by the FTC.

(b) As we're given the interval, we need only determine which curve is larger and on which intervals. As $0 \leq \cos x \leq 1$ on $[0, \pi/2]$, we see that

$$0 \leq (\cos x)^2 \leq \cos x$$

on $[0, \pi/2]$. Since $\cos x \leq 0$ for $x \in [\pi/2, \pi]$, we see that

$$\cos x \leq 0 \leq (\cos x)^2$$

on $[\pi/2, \pi]$. Hence, the area is given

$$\int_0^{\pi/2} \cos x - (\cos x)^2 dx + \int_{\pi/2}^{\pi} (\cos x)^2 - \cos x dx.$$

To integrate $(\cos x)^2$, we need to apply a trig identity which we get from the double angle identity for $\cos x$:

$$\cos 2x = (\cos x)^2 - (\sin x)^2 = (\cos x)^2 - (1 - (\cos x)^2) = 2(\cos x)^2 - 1 \Longrightarrow (\cos x)^2 = \frac{1 + \cos 2x}{2}.$$

Thus,

$$\int (\cos x)^2 dx = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right),$$

this can be seen with the substitution $u = 2x$ (try this yourself). Therefore, the area is

$$\begin{aligned}
\int_0^{\pi/2} (\cos x) - (\cos x)^2 dx - \int_{\pi/2}^{\pi} (\cos x) - (\cos x)^2 dx &= \sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x|_0^{\pi/2} \\
&\quad - \left(\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x|_{\pi/2}^{\pi}\right) \\
&= 1 - \frac{\pi}{4} - \frac{1}{4}(0) - \left(0 - \frac{1}{2}(0) - \frac{1}{4}(0)\right) \\
&\quad - \left(0 - \frac{\pi}{2} - \frac{1}{4}(0) - \left(1 - \frac{\pi}{4} - \frac{1}{4}(0)\right)\right) \\
&= 1 - \pi/4 - (-\pi/2 - 1 + \pi/4) \\
&= \boxed{2.}
\end{aligned}$$

(c) Note that

$$e^x = e^{-x} \implies x = -x \implies x = 0$$

and $e^1 > e^{-1}$ so $e^x \geq e^{-x}$ on $[0,1]$ while $e^{-x} \geq e^x$ on $[-1,0]$. Hence, the area is given by

$$\begin{aligned}\int_{-1}^{0} e^{-x} - e^x dx + \int_0^1 e^x - e^{-x} dx &= -e^{-x} - e^x\Big|_{-1}^{0} + e^x + e^{-x}\Big|_0^1 \\ &= -e^0 - e^0 - \left(-e^1 - e^{-1}\right) + e^1 + e^{-1} - \left(e^0 + e^0\right) \\ &= -2 + 2e + 2e - 2 \\ &= \boxed{4e - 4.}\end{aligned}$$