Exercise 1. Evaluate:

(a) $\int x^2(1-x)^{100}dx$

(b) $\int \sin x \,(\cos x)^3\, dx$

(c) $\int \dfrac{\ln x}{x}dx$

(d) $\int (\sin x)^3 dx$

(e) [Challenge]: Generalizing (d), evaluate $\int (\sin x)^{2k+1}dx$ for k a non-negative integer

Solutions:

(a) Let's take $u = 1 - x$ so $x = 1 - u$ and

$$du = -dx \Longrightarrow dx = -du.$$

This u is the only "inside" function which doesn't result in a trivial substitution, that is, $u = x$. So, this gives us:

$$\begin{aligned}\int x^2(1-x)^{100}dx &= \int x^2u^{100}(-du) && \text{[using our substitution]}\\ &= -\int (1-u)^2u^{100}du. && \text{[as } x = 1-u\text{]}\end{aligned}$$

As our integral is now *only* in terms of u, we may evaluate thinking of u as our variable. Note that

$$-\int (1-u)^2u^{100}du = -\int (1-2u+u^2)u^{100}du = \int -u^{100}+2u^{101}-u^{102}du = -\frac{u^{101}}{101}+\frac{2u^{102}}{102}-\frac{u^{103}}{103}+C.$$

Now, we just need to undo our substitution, i.e. replace u with $1 - x$, to get

$$\int x^2(1-x)^{100}dx = -\frac{u^{101}}{101}+\frac{2u^{102}}{102}-\frac{u^{103}}{103}+C = \boxed{-\frac{(1-x)^{101}}{101}+\frac{2(1-x)^{102}}{102}-\frac{(1-x)^{103}}{103}+C.}$$

(b) Note $\frac{d}{dx}\cos x = -\sin x$ which appears in our integrand so let's take $u = \cos x$ then

$$du = -\sin x dx \Longrightarrow dx = -\frac{du}{\sin x}.$$

Now, we have

$$\begin{aligned}\int \sin x\,(\cos x)^3\,dx &= \int \sin x u^3\left(-\frac{du}{\sin x}\right) && \text{[using our substitution]}\\ &= -\int u^3du\\ &= -\frac{u^4}{4}+C && \text{[by the anti-power rule]}\\ &= \boxed{-\frac{(\cos x)^4}{4}+C.} && \text{[as } u = \cos x\text{]}\end{aligned}$$

(c) Note that $\frac{d}{dx}\ln x = 1/x$ appears in our integrand so let's take $u = \ln x$ then

$$du = \frac{1}{x}dx \Longrightarrow dx = xdu.$$

Now, we have

$$\begin{aligned}
\int \frac{\ln x}{x}dx &= \int \frac{u}{x}(xdu) && \text{[using our substitution]} \\
&= \int udu \\
&= \frac{u^2}{2} + C && \text{[by the anti-power rule]} \\
&= \boxed{\frac{(\ln x)^2}{2} + C.} && \text{[as } u = \ln x\text{]}
\end{aligned}$$

(d) This integral is a little tricky as the obvious choice for your "inside" function (i.e. $u = \sin x$) does not lead to cancellation with the resulting dx, that is, $dx = du/\cos x$. So, we'll apply some trig identities to get a nicer integral to work with, namely we'll use the fact that $(\sin x)^2 = 1 - (\cos x)^2$. Therefore, we have

$$\begin{aligned}
\int (\sin x)^3 dx &= \int \sin x(\sin x)^2 dx \\
&= \int \sin x \left(1 - (\cos x)^2\right) dx \\
&= \int \sin x dx - \int \sin x(\cos x)^2 dx \\
&= -\cos x - \int \sin x(\cos x)^2 dx.
\end{aligned}$$

Now, just like in part (b), let's take $u = \cos x$ then

$$du = -\sin x dx \Longrightarrow dx = -\frac{du}{\sin x}.$$

Hence,

$$\int \sin x(\cos x)^2 dx = \int (\sin x)u^2 \left(-\frac{du}{\sin x}\right) = -\int u^2 du = -\frac{u^3}{3} + C = -\frac{(\cos x)^3}{3} + C$$

which, when combined with the above, tells us that

$$\int (\sin x)^3 dx = \boxed{-\cos x + \frac{(\cos x)^3}{3} + C.}$$

(e) Let's use the same trick as in part (d) to rewrite:

$$\int (\sin x)^{2k+1} dx = \int \sin x(\sin x)^{2k} dx = \int \sin x(1 - (\cos x)^2)^k dx.$$

Now, let's take $u = \cos x$ then

$$du = -\sin x dx \Longrightarrow dx = -\frac{du}{\sin x}.$$

So, we have

$$\begin{aligned}\int (\sin x)^{2k+1} dx &= \int \sin x (1-(\cos x)^2)^k dx \\ &= \int \sin x (1-u^2)^k \left(-\frac{du}{\sin x}\right) \\ &= -\int (1-u^2)^k du.\end{aligned}$$

Now, we can use the Binomial Theorem to get

$$\begin{aligned}\int (\sin x)^{2k+1} dx &= -\int (1-u^2)^k du \\ &= -\int \sum_{i=0}^{k} \binom{k}{i} (-1)^i u^{2i} du \\ &= -\sum_{i=0}^{k} (-1)^i \binom{k}{i} \int u^{2i} du \\ &= \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} \frac{u^{2i+1}}{2i+1} + C \\ &= \boxed{\sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} \frac{(\cos x)^{2i+1}}{2i+1} + C}\end{aligned}$$

where

$$\binom{k}{i} = \frac{k!}{i!(k-i)!}$$

is a binomial coefficient.

Let's verify for $k = 1$ to see if our answer agrees with the one we found in part (d). The above tells us that

$$\int (\sin x)^{2(1)+1} dx = \int (\sin x)^3 dx = \sum_{i=0}^{1} (-1)^{i+1} \binom{1}{i} \frac{(\cos x)^{2i+1}}{2i+1} + C = -\cos x + \frac{(\cos x)^3}{3} + C$$

just as we got in part (d).

Exercise 2. Evaluate:

(a) $\displaystyle\int_0^{\pi/4} (\sec x)^2 e^{\tan x} dx$

(b) $\displaystyle\int_0^1 x\sqrt{3x+1}dx$

Solutions:

(a) Note that $\frac{d}{dx}\tan x = (\sec x)^2$ which appear in our integrand so let's take $u = \tan x$ then

$$du = (\sec x)^2 dx \Longrightarrow dx = \frac{du}{(\sec x)^2}.$$

And, we know that our bounds will become $u(0) = 0$ and $u(\pi/4) = 1$ after our substitution. So, we have

$$\begin{aligned}
\int_0^{\pi/4} (\sec x)^2 e^{\tan x} dx &= \int_0^1 (\sec x)^2 e^u \left(\frac{du}{(\sec x)^2}\right) && \text{[using our substitution]} \\
&= \int_0^1 e^u du \\
&= e^u|_0^1 && \text{[by the FTC]} \\
&= e^1 - e^0 = \boxed{e - 1.}
\end{aligned}$$

(b) The only "inside" function that we have in our integrand is $3x + 1$ so let's take $u = 3x + 1$ then

$$du = 3dx \Longrightarrow dx = \frac{du}{3}.$$

And, we know that our bounds will become $u(0) = 1$ and $u(1) = 4$. So, we have

$$\int_0^1 x\sqrt{3x+1}dx = \int_1^4 x\sqrt{u}\left(\frac{du}{3}\right) = \frac{1}{3}\int_1^4 x\sqrt{u}du$$

but we need our new integrand to be *only* in terms of u. So, we need to solve for x in $u = 3x + 1$ then substitute that back in:

$$u = 3x + 1 \Longrightarrow x = \frac{u-1}{3}.$$

Hence, we have

$$\begin{aligned}
\int_0^1 x\sqrt{3x+1}dx &= \frac{1}{3}\int_1^4 x\sqrt{u}du \\
&= \frac{1}{3}\int_1^4 \left(\frac{u-1}{3}\right)\sqrt{u}du \\
&= \frac{1}{9}\int_1^4 u^{3/2} - u^{1/2}du \\
&= \frac{1}{9}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right)\Bigg|_1^4 \\
&= \boxed{\frac{1}{9}\left(\frac{2^6}{5} - \frac{2^4}{3} - \frac{2}{5} + \frac{2}{3}\right).}
\end{aligned}$$

Exercise 3. Find the average of $f(x) = \frac{1}{x \ln x}$ on the interval $[e, e^5]$.

Solution: The average of $f(x)$ on the interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

So, for use, this problem is asked us to compute

$$\frac{1}{e^5 - e} \int_e^{e^5} \frac{1}{x \ln x} dx.$$

Note that $\frac{d}{dx} \ln x = 1/x$ which appears in our integrand so let's take $u = \ln x$ then

$$du = \frac{1}{x} dx \Longrightarrow dx = x du.$$

And, we know that our bounds become $u(e) = 1$ and $u(e^5) = 5$ Hence, we have

$$\begin{aligned}
\frac{1}{e^5 - e} \int_e^{e^5} \frac{1}{x \ln x} dx &= \frac{1}{e^5 - e} \int_1^5 \frac{1}{xu} (x du) && \text{[using our substitution]} \\
&= \frac{1}{e^5 - e} \int_1^5 \frac{1}{u} du \\
&= \frac{1}{e^5 - e} \ln u \Big|_1^5 && \text{[by the FTC]} \\
&= \boxed{\frac{\ln 5}{e^5 - e}.}
\end{aligned}$$