Exercise 1. Evaluate:

(a) $\int \sqrt{t} - \frac{4}{\sqrt{t}} dt$

(b) $\int_{-1}^{1} e^t + t^3 - 2\pi t dt$

(c) $\int_0^{\pi} 3\cos x - \pi \sin x dx$

Solutions:

(a) For this we use the anti-power rule,

$$\int \sqrt{t} - \frac{4}{\sqrt{t}} dt = \int t^{1/2} - 4t^{-1/2} dt = \boxed{\frac{t^{3/2}}{3/2} - \frac{4t^{1/2}}{1/2} + C.}$$

(b) By the fundamental theorem of calculus, we need only find an antiderivative of $e^t + t^3 - 2\pi t$ to solve this definite integral. Using the fact that $\int e^t dt = e^t + C$ and the anti-power rule, we see that

$$\int e^t + t^3 - 2\pi t dt = e^t + \frac{t^4}{4} - \pi t^2 + C.$$

Hence, we have that

$$\int_{-1}^{1} e^t + t^3 - 2\pi t dt = e^t + \frac{t^4}{4} - \pi t^2 \Big|_{-1}^{1} = \boxed{e + 1/4 - \pi - \left(e^{-1} + 1/4 - \pi\right)}$$

by the fundamental theorem of calculus.

(c) Just as with (1b), we need only find an antiderivative of $3\cos x - \pi \sin x$ to evaluate this definite integral. Using the fact that $\int \sin x dx = -\cos x + C$ and $\int \cos x dx = \sin x + C$, we see that

$$\int_0^{\pi} 3\cos x - \pi \sin x dx = 3\sin x + \pi \cos x|_0^{\pi} = \boxed{3(0) + \pi(-1) - (3(0) + \pi(1)).}$$

Exercise 2. Suppose that

$$\int_0^1 f(x)dx = -3 \quad \text{and} \quad \int_0^1 g(x)dx = 2.$$

Find the exact values of

(a) $\int_0^1 (4f(x) - 3g(x))dx$

(b) $\int_0^{1/3} f(x)dx + \int_{1/3}^1 f(x)dx$

Solutions:

(a) As $\int$ is linear, we may distribute it over addition and pull out constants so

$$\int_0^1 (4f(x) - 3g(x))dx = 4\int_0^1 f(x)dx - 3\int_0^1 g(x)dx = \boxed{4(-3) - 3(2).}$$

(b) Since $\int_a^b f(x)dx$ is the (signed) area of $f(x)$ from a to b, we see that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

for $a < c < b$ as the RHS is the (signed) area under $f(x)$ from a to c plus the (signed) area from c to b which is the total (signed) area of $f(x)$ from a to b. Therefore,

$$\int_0^{1/3} f(x)dx + \int_{1/3}^1 f(x)dx = \int_0^1 f(x)dx = \boxed{-3.}$$

Exercise 3. Compute:

(a) $\dfrac{d}{dx}\displaystyle\int_{1/x}^{\cos x} 3e^{(t^2)}dt$

(b) $\dfrac{d}{dt}\displaystyle\int_{-t^2}^{1} \frac{1}{1+x^2}dx$

Solutions: Recall the fundamental theorem of calculus tells us that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = f(b(t))b'(t) - f(a(t))a'(t)$$

which we'll use for these problems.

(a) By the fundamental theorem of calculus:

$$\frac{d}{dx}\int_{1/x}^{\cos x} 3e^{(t^2)}dt = 3e^{(\cos x)^2}\frac{d}{dx}(\cos x) - 3e^{(1/x)^2}\frac{d}{dx}\left(\frac{1}{x}\right) = \boxed{-3e^{(\cos x)^2}(\sin x) + \frac{3e^{1/x^2}}{x^2}}.$$

(b) By the fundamental theorem of calculus:

$$\frac{d}{dt}\int_{-t^2}^{1}\frac{1}{1+x^2}dx = \frac{1}{1+1^2}\frac{d}{dt}(1) - \frac{1}{1+(-t^2)^2}\frac{d}{dt}(-t^2) = \boxed{\frac{2t}{1+t^4}}.$$

Exercise 4. Find the total area of $f(x) = x^2 - 1$ on $[-1, 2]$.

Solution: The total area of a function on an interval $[a, b]$ is defined to be the unsigned area between the function and the x-axis so it's given by

$$\int_a^b |f(x)| dx.$$

For us, this problem becomes computing

$$\int_{-1}^2 |x^2 - 1| dx.$$

We want to find when our function is negative and when it's negative. Note that

$$f(x) \geq 0 \iff x^2 - 1 \geq 0 \iff x^2 \geq 1 \iff |x| \geq 1$$

so $f(x)$ is positive on the intervals $(-\infty, -1)$ and $(1, \infty)$. So, we split our integral at 1 and give the portion on $(-1, 1)$ a negative:

$$\begin{aligned} \int_{-1}^2 |x^2 - 1| dx &= -\int_{-1}^1 x^2 - 1 dx + \int_1^2 x^2 - 1 dx \\ &= x - \frac{x^3}{3}\Big|_{-1}^1 + \frac{x^3}{3} - x\Big|_1^2 \\ &= \boxed{1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) + \frac{8}{3} - 2 - \left(\frac{1}{3} - 1\right).} \end{aligned}$$

Exercise 5. Evaluate $\int_{-2}^{2} \sqrt{4-x^2}dx$.

Solution: As we don't currently have the tools to compute $\int \sqrt{4-x^2}dx$, we need to think geometrically to evaluate this definite integral. Note that

$$y^2 + x^2 = 2^2$$

is the equation of the circle of radius 2 centered at the origin. Solving for y, we see that

$$y = \pm\sqrt{4-x^2}$$

so $\sqrt{4-x^2}$ corresponds to the upper half of the circle of radius 2 centered at the origin. Thus, the area between the x-axis and $\sqrt{4-x^2}$ on the interval $[-2, 2]$ is

$$\int_{-2}^{2} \sqrt{4-x^2}dx = \frac{1}{2}(\pi(2^2)) = \boxed{2\pi.}$$

Exercise 6. Find the function $f(x)$ such that $f(\pi) = 0$ and

$$f'(x) = 9\cos x + 5\sin x.$$

Solution: We know that

$$\int 9\cos x + 5\sin x dx = 9\sin x - 5\cos x + C$$

so $f(x) = 9\sin x - 5\cos x + C$ for some constant C. But, given that $f(\pi) = 0$, we can determine this C exactly:

$$0 = f(\pi) = 9\sin\pi - 5\cos\pi + C = 5 + C \Longrightarrow C = -5.$$

Thus, we conclude that

$$f(x) = \boxed{9\sin x - 5\cos x - 5.}$$