Exercise 1. Given $\sum_{i=1}^{10} x_i = 4$ and $\sum_{i=1}^{10} y_i = 7$, find exact values for:

(a) $\sum_{i=1}^{10}(4x_i + 3y_i - 1)$

(b) $\sum_{j=3}^{12} x_{j-2}$

(c) $\sum_{i=1}^{5} x_i + \sum_{i=6}^{10} x_i$

Solutions:

(a) Using linearity, we see that

$$\sum_{i=1}^{10}(4x_i + 3y_i - 1) = 4\sum_{i=1}^{10} x_i + 3\sum_{i=1}^{10} y_i - \sum_{i=1}^{10} = 4(4) + 3(7) - 10 = \boxed{27.}$$

(b) Writing it out we see

$$\sum_{j=3}^{12} x_{j-2} = x_{3-2} + x_{4-2} + x_{5-2} + \cdots + x_{12-2} = x_1 + x_2 + x_3 + \cdots + x_{10} = \boxed{4.}$$

(c) Writing it out we see

$$\sum_{i=1}^{5} x_i + \sum_{i=6}^{10} x_i = x_1 + x_2 + \cdots + x_5 + x_6 + x_7 + \cdots + x_{10} = \boxed{4.}$$

Exercise 2. Compute L_4 and R_4 for:

(a) $f(x) = \frac{x}{x^2+1}$ on $[1, 3]$

(b) $g(x) = \sin x$ on $[0, \pi]$

solutions:

(a) First, we see that $x_0 = 1$ and $\Delta x = \frac{3-1}{4} = 1/2$ so $x_i = 1 + i/2$. Hence, we have

$$\begin{aligned} L_4 &= \sum_{i=1}^{4} f(x_{i-1})\Delta x \\ &= \sum_{i=1}^{4} \frac{1}{2} f(1 + (i-1)/2) \\ &= \boxed{\frac{1}{2}\left(\frac{1}{1^2+1}\right) + \frac{1}{2}\left(\frac{1+1/2}{(1+1/2)^2+1}\right) + \frac{1}{2}\left(\frac{2}{2^2+1}\right) + \frac{1}{2}\left(\frac{1+3/2}{(1+3/2)^2+1}\right)} \end{aligned}$$

and

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} f(x_i)\Delta x \\ &= \sum_{i=1}^{4} \frac{1}{2} f(1 + i/2) \\ &= \boxed{\frac{1}{2}\left(\frac{1+1/2}{(1+1/2)^2+1}\right) + \frac{1}{2}\left(\frac{2}{2^2+1}\right) + \frac{1}{2}\left(\frac{1+3/2}{(1+3/2)^2-1}\right) + \frac{1}{2}\left(\frac{3}{3^2+1}\right).} \end{aligned}$$

(b) First, we have $x_0 = 0$ and $\Delta x = \frac{\pi - 0}{4} = \pi/4$ so $x_i = i\pi/4$. Hence, we have

$$\begin{aligned} L_4 &= \sum_{i=1}^{4} g(x_{i-1})\Delta x \\ &= \sum_{i=1}^{4} \frac{\pi}{4} g\left(\frac{(i-1)\pi}{4}\right) \\ &= \boxed{\frac{\pi}{4}(\sin 0) + \frac{\pi}{4}(\sin \pi/4) + \frac{\pi}{4}(\sin \pi/2) + \frac{\pi}{4}(\sin 3\pi/2)} \end{aligned}$$

and

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} g(x_i)\Delta x \\ &= \sum_{i=1}^{4} \frac{\pi}{4}(\sin i\pi/4) \\ &= \boxed{\frac{\pi}{4}(\sin \pi/4) + \frac{\pi}{4}(\sin \pi/2) + \frac{\pi}{4}(\sin 3\pi/4) + \frac{\pi}{4}(\sin \pi).} \end{aligned}$$

Exercise 3. Evaluate:

(a) $\int_{-4}^{4} \sqrt{16 - x^2} dx$

(b) $\int_{-3}^{1} 2x + 4dx$

(c) (Challenge) $\int_{0}^{1} x^2 dx$

[*Hint: Take the limit of the right endpoint approximation using the fact that* $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$]

Solutions:

1. Let $f(x) = \sqrt{16 - x^2}$ then, we see that

$$f(x)^2 + x^2 = 16 - x^2 + x^2 = 4^2,$$

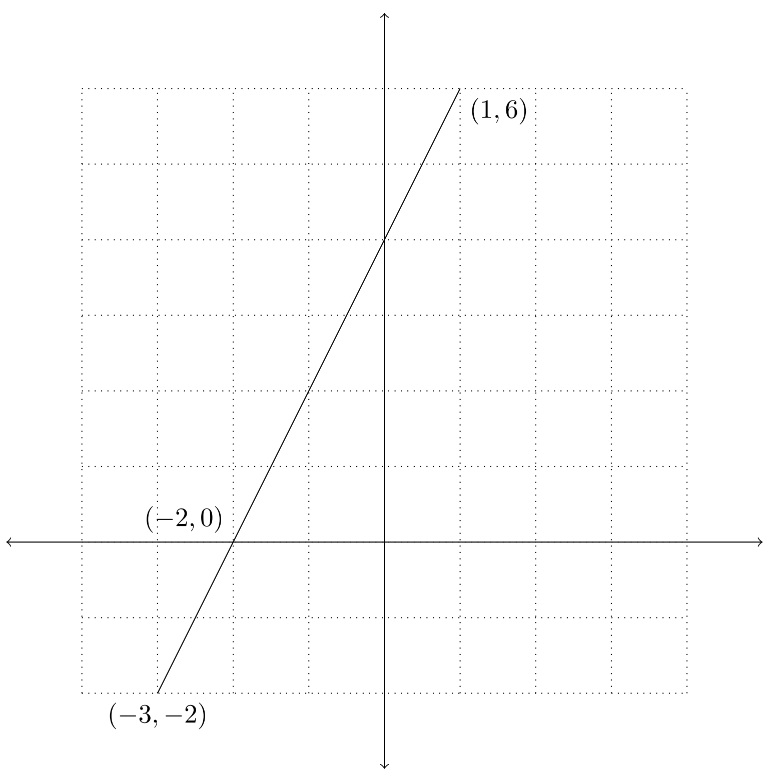
that is, $f(x)$ satisfies the equation for the circle centered at the origin of radius 4. Hence, $f(x)$ lies on some portion of this circle. Note that $f(x)$ is never negative as square root is never negative so it must be contained in the upper half of the circle. Moreover, $(4, f(4)) = (4, 0)$ and $(-4, f(-4)) = (-4, 0)$ so $f(x)$ must be the entire upper half of this circle by the intermediate value theorem.

[Note: In general, $\sqrt{r^2 - x^2}$ *for* $-r \leq x \leq r$*, will always be the upper half of the circle centered at the origin of radius* r*. Try graphing it!]*

We know that the area of this circle is $\pi(4)^2 = 16\pi$ but we only want the area of the upper half, so we divide by 2 to get:

$$\int_{-4}^{4} \sqrt{16 - x^2} dx = \frac{16\pi}{2} = \boxed{8\pi.}$$

2. Graphing $2x + 4$ for $-3 \leq x \leq 1$, we see:



Thus, we have

$$\int_{-3}^{1} 2x + 4dx = \frac{1}{2}(3)(6) - \frac{1}{2}(1)(2) = 9 - 1 = \boxed{8.}$$

3. Let's compute the n-th right endpoint approximation for $f(x) = x^2$ on $[0, 1]$. First, we see that $x_0 = 0$ and $\Delta x = \frac{1-0}{n} = 1/n$ so $x_i = i/n$. Now, we see that

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}$$

where the final equality follows from the hint. If we take the limit as n tends towards infinity, our approximation of the area under x^2 on $[0, 1]$ gets arbitrarily close to the true value. That means,

$$\int_{0}^{1} x^2 dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

where we evaluated the limit using the dominant power rule.