Exercise 1. (Derivatives of e^x and $\ln x$)
Compute $f'(x)$ where:

(a) $f(x) = (\sin x)^{x^2}$

(b) $f(x) = \ln\left(\tan x + e^{-2x}\right)$

(c) $f(x) = \left(x^2 - 1\right)^{\ln x}$

(d) $f(x) = \ln\left((\cos x)^{2x^3}\right)$

Solution:

(a) Write

$$f(x) = e^{x^2 \ln(\sin x)}$$

then

$$\begin{aligned} f'(x) &= e^{x^2 \ln(\sin x)} \frac{d}{dx}\left(x^2 \ln(\sin x)\right) && \text{[derivative of } e^x \text{ and chain rule]} \\ &= (\sin x)^{x^2}\left(2x\ln(\sin x) + x^2 \frac{d}{dx}\left(\ln(\sin x)\right)\right) && \text{[product and power rule]} \\ &= \boxed{(\sin x)^{x^2}\left(2x\ln(\sin x) + \frac{x^2 \cos x}{\sin x}\right).} && \text{[derivative of } \ln x\text{]} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{1}{\tan x + e^{-2x}} \cdot \frac{d}{dx}\left(\tan x + e^{-2x}\right) && \text{[derivative of } \ln x \text{ and chain rule]} \\ &= \boxed{\frac{(\sec x)^2 - 2e^{-2x}}{\tan x + e^{-2x}}.} && \text{[derivative of } e^x \text{ and chain rule]} \end{aligned}$$

(c) Write

$$f(x) = e^{(\ln x)\ln(x^2-1)}$$

then

$$\begin{aligned} f'(x) &= e^{(\ln x)\ln(x^2-1)} \frac{d}{dx}\left((\ln x)\ln(x^2-1)\right) && \text{[derivative of } e^x \text{ and chain rule]} \\ &= \boxed{(x^2-1)^{\ln x}\left(\frac{\ln(x^2-1)}{x} + \frac{2x(\ln x)}{x^2-1}\right).} && \text{[derivative of } \ln x\text{, product and chain rule]} \end{aligned}$$

(d) Note

$$f'(x) = \frac{1}{(\cos x)^{2x^3}} \cdot \frac{d}{dx}\left((\cos x)^{2x^3}\right) \qquad \text{[derivative of } \ln x \text{ and chain rule]}$$

so we need to compute the derivative of $(\cos x)^{2x^3}$. Write

$$(\cos x)^{2x^3} = e^{2x^3 \ln(\cos x)}$$

then

$$\frac{d}{dx}\left((\cos x)^{2x^3}\right) = e^{2x^3\ln(\cos x)}\frac{d}{dx}\left(2x^3\ln(\cos x)\right) \qquad \text{[derivative of } e^x \text{ and chain rule]}$$
$$= (\cos x)^{2x^3}\left(6x^2\ln(\cos x) - \frac{2x^3\sin x}{\cos x}\right). \qquad \text{[derivative of } \ln x\text{, product and chain rule]}$$

So, we conclude that

$$f'(x) = \frac{1}{(\cos x)^{2x^3}}(\cos x)^{2x^3}\left(6x^2\ln(\cos x) - \frac{2x^3\sin x}{\cos x}\right) = \boxed{6x^2\ln(\cos x) - \frac{2x^3\sin x}{\cos x}.}$$

Exercise 2. (Linear approximations)
Use linear approximations to approximate the following values:

(a) $\ln\left(e^3 + \frac{1}{100}\right)$

(b) $\sin 1$ (*Hint:* $\pi/3 \approx 1$)

(c) $\sqrt{99}$

(d) $\ln\left(\frac{3}{\pi}\right)$ (*Hint:* $\pi/3 \approx 1$)

Solution:

(a) Consider $f(x) = \ln x$ then we'll approximate this function at $x = e^3$ as e^3 is close to $e^3 + 1/100$ and we know $f(e^3) = 3$. Since $f'(x) = 1/x$, the linear approximation of $f(x)$ at $x = e^3$ is

$$L(x) = f(e^3) + f'(e^3)(x - e^3) = 3 + \frac{x - e^3}{e^3}.$$

Hence,

$$f\left(e^3 + \frac{1}{100}\right) \approx L\left(e^3 + \frac{1}{100}\right) = \boxed{3 + \frac{1}{100e^3}.}$$

(b) Consider $f(x) = \sin x$ then we'll approximate this function at $x = \pi/3$ as $\pi/3$ is close to 1 and we know $f(\pi/3) = \sqrt{3}/2$. Since $f'(x) = \cos x$, the linear approximation of $f(x)$ at $x = \pi/3$ is

$$L(x) = f(\pi/3) + f'(\pi/3)(x - \pi/3) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \pi/3).$$

Hence,

$$f(1) \approx L(1) = \boxed{\frac{\sqrt{3}}{2} + \frac{1}{2}(1 - \pi/3).}$$

(c) Consider $f(x) = \sqrt{x}$ then we'll approximate this function at $x = 100$ as 100 is close to 99 and we know $f(100) = 10$. Since $f'(x) = \frac{1}{2}x^{-1/2}$, the linear approximation of $f(x)$ at $x = 100$ is

$$L(x) = f(100) + f'(100)(x - 100) = 10 + \frac{1}{2\sqrt{100}}(x - 100) = 10 + \frac{1}{20}(x - 100).$$

Hence,

$$f(99) \approx L(100) = \boxed{10 - \frac{1}{20}.}$$

(d) Consider $f(x) = \ln x$ then we'll approximate this function at $x = 1$ as 1 is close to $3/\pi$ and we know that $f(1) = 0$. Since $f'(x) = 1/x$, the linear approximation of $f(x)$ at $x = 1$ is

$$L(x) = f(1) + f'(1)(x - 1) = x - 1.$$

Hence,

$$f(3/\pi) \approx L(3/\pi) = \boxed{\frac{3}{\pi} - 1.}$$

Exercise 3. (Minima and Maxima)
Find the local minima and maxima of $f(x)$ on the given interval:

(a) $f(x) = x + \sin x$ over $[0, 2\pi]$

(b) $f(x) = \frac{x}{1+x^2}$ over $[0, 10]$

(c) $f(x) = 3x\sqrt{1 - x^2}$ over $[0, 1]$

(d) $f(x) = x^2 + \frac{4}{x}$ over $[1, 4]$

Solution:

(a) First, we compute the critical points of $f(x)$ on $[0, 2\pi]$. Note $f'(x) = 1 + \cos x$ so

$$0 = f'(x) \iff \cos x = -1$$

which implies that $x = \pi$. Now, we need only compare the values of $f(x)$ at $0, \pi, 2\pi$:

x	0	π	2π
$f(x)$	0	π	2π

Note that $f'(\pi/2) > 0$ and $f'(3\pi/2) > 0$ so $x = \pi$ is not a local minima or maxima of $f(x)$. Then the minimum and maximum of $f(x)$ on $[0, 2\pi]$ are $\boxed{0}$ and $\boxed{2\pi}$, respectively.

(b) First, we compute the critical points of $f(x)$ on $[0, 10]$. By the quotient rule,

$$f'(x) = \frac{1 + x^2 - x(2x)}{(1 + x^2)^2}$$

so

$$0 = f'(x) \iff 0 = 1 + x^2 - 2x^2 = 1 - x^2$$

which implies $x = 1$. Now, we need only compare the values of $f(x)$ at $0, 1, 10$:

x	0	1	10
$f(x)$	0	$1/2$	$10/101$

Since $10/101 < 10/100 = 1/10 < 1/2$ and $0 < 1/2$, $1/2$ is an absolute maximum of $f(x)$ at $x = 1$ on $[0, 10]$ so we need not check the sign of $f'(x)$ around $x = 1$. Note that $f'(2) < 0$ so $f(10) = 10/101$ is a local minimum. Hence, the local minima of $f(x)$ on $[0, 10]$ are $\boxed{0, 10/101}$ and the local maximum of $f(x)$ on $[0, 10]$ is $\boxed{1/2}$.

(c) First, we compute the critical points of $f(x)$ on $[0, 1]$. Note that

$$f'(x) = 3\sqrt{1 - x^2} + \frac{3x(-2x)}{2\sqrt{1 - x^2}} = 3\sqrt{1 - x^2} - \frac{3x^2}{\sqrt{1 - x^2}}$$

which is undefined at $x = 1$ so $x = 1$ is a critical point. Now, multiplying both sides of $0 = f'(x)$ by $\sqrt{1 - x^2}$ when $x < 1$, we have

$$0 = f'(x) \iff 0 = 3(1 - x^2) - 3x^2 = 3(1 - 2x^2)$$

which implies $x = \sqrt{1/2}$. Now, we need only compare the values of $f(x)$ at $0, \sqrt{1/2}, 1$:

x	0	$\sqrt{1/2}$	1
$f(x)$	0	$3/2$	0

Hence, the local minimum of $f(x)$ on $[0, 1]$ is $\boxed{1}$ and the local maximum of $f(x)$ on $[0, 1]$ is $\boxed{3/2.}$

(d) First, we compute the critical points of $f(x)$ on $[1, 4]$. Note that $f'(x) = 2x - 4x^{-2}$ so

$$0 = f'(x) \iff 0 = 2x^3 - 4 \iff x^3 = 2$$

which implies $x = \sqrt[3]{2}$. Now, we need only compare the values of $f(x)$ at $1, \sqrt[3]{2}, 4$:

x	1	$\sqrt[3]{2}$	4
$f(x)$	5	$2^{2/3} + 4(2^{-1/3})$	17

Since

$$f'(\sqrt[3]{3/2}) = \frac{2\left(\sqrt[3]{3/2}\right)^3 - 4}{\left(\sqrt[3]{3/2}\right)^2} = \frac{-1}{\left(\sqrt[3]{3/2}\right)^2} < 0$$

and

$$f'(\sqrt[3]{3}) = \frac{2\left(\sqrt[3]{3}\right)^3 - 4}{\left(\sqrt[3]{3}\right)} = \frac{2}{\left(\sqrt[3]{3}\right)^2} > 0,$$

we see that $f(\sqrt[3]{2})$ is a local minimum. So, in conclusion, we see that the local minimum of $f(x)$ on $[1, 4]$ is $\boxed{2^{2/3} + 4(2^{-1/3})}$ and the local maxima of $f(x)$ on $[1, 4]$ are $\boxed{5, 17}$.

Exercise 4. (Mean Value Theorem)
Use the mean value theorem and find all $1 < c < 2$ such that $f(2) - f(1) = f'(c)(2 - 1)$:

(a) $f(x) = x^3$

(b) $f(x) = \sin(\pi x)$

(c) $f(x) = 1 + x + x^2 + x^3$

Solution: Note that all $f(x)$ are continuous on the interval $[1, 2]$ and differentiable on (a, b) so the mean value theorem applies in all cases and ensures the existence of such c.

(a) Note that $f'(x) = 3x^2$ and $f(1) = 1$ and $f(2) = 8$. So, we want $1 < c < 2$ such that

$$8 - 1 = 3c^2(2 - 1) \iff c^2 = \frac{7}{3}$$

which implies that $\boxed{c = \sqrt{7/3}.}$

(b) Note that $f'(x) = \pi \cos(\pi x)$ by the chain rule and $f(1) = 0 = f(2)$. So, we want $1 < c < 2$ such that

$$0 - 0 = \pi \cos(\pi c)(2 - 1) \iff \cos(\pi c) = 0$$

which implies that $\boxed{c = 3\pi/2.}$

(c) Note that $f'(x) = 1 + 2x + 3x^2$ and $f(1) = 4$ and $f(2) = 15$. So, we want $1 < c < 2$ such that

$$15 - 4 = (1 + 2c + 3c^2)(2 - 1) \iff 3c^2 + 2c - 11 = 0 \iff c = \frac{-2 \pm \sqrt{4 - 4(3)(-11)}}{2(3)} = \frac{-1 \pm \sqrt{34}}{3}$$

which implies $\boxed{c = \frac{-1 + \sqrt{34}}{3}.}$