

Exercise 1. (Trigonometry)

- (a) Find all solutions in $[0, 2\pi]$ to $\cos(2\theta) = \sin \theta$.
- (b) Compute the exact value of $\sin\left(\frac{11\pi}{12}\right)$.
- (c) Compute the exact value of $\sin^{-1}(-1/2)$.

Solution:

- (a) Using the angle-sum identity for cos and Pythagorean identity, we may write

$$\cos(2\theta) = (\cos \theta)^2 - (\sin \theta)^2 = 1 - 2(\sin \theta)^2.$$

Hence, we have

$$\begin{aligned} \cos(2\theta) = \sin \theta &\iff 1 - 2(\sin \theta)^2 = \sin \theta \\ &\iff 2(\sin \theta)^2 + \sin \theta - 1 = 0 \\ &\iff \sin \theta = \frac{-1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm 3}{4}. \end{aligned}$$

So, $\sin \theta = -1$ or $\sin \theta = 1/2$. When $\sin \theta = -1$, we know that $\theta = 3\pi/2$. When $\sin \theta = 1/2$, we know that $\theta = \pi/6$ or $\theta = 5\pi/6$. Therefore, we conclude that

$$\boxed{\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}}$$

are all such solutions.

- (b) We want to write $11\pi/12$ as the sum of known values of cos and sin. Since $12 = 3 \cdot 4$ and we know the exact values of cos and sin at integer multiples of $\pi/4$ and $\pi/3$, we need to find integers a, b such that

$$\frac{11\pi}{12} = a\frac{\pi}{4} + b\frac{\pi}{3} = \frac{3a + 4b}{12}\pi \iff 11 = 3a + 4b.$$

Since $11 - 8 = 3$, we may take $a = 1$ and $b = 2$. Now, using the angle-sum identity for sin, we have

$$\begin{aligned} \sin\left(\frac{11\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) \\ &= \boxed{\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}. \end{aligned}$$

- (c) If $\theta = \sin^{-1}(-1/2)$ then

$$\sin \theta = -\frac{1}{2}.$$

So, we want to find such θ in the interval $[-\pi/2, \pi/2]$. Referencing a unit circle, we see that $-\pi/6$ is the only value that works. Thus,

$$\sin^{-1}(-1/2) = \boxed{-\frac{\pi}{6}}.$$

Exercise 2. (One-sided limits)

Find all values, if they exist, of k which make $f(x)$ continuous and k which make $f(x)$ differentiable.

(a) Let

$$f(x) = \begin{cases} k\sqrt{x+4} + k^2 & x \geq 0 \\ k^5x - 1 & x < 0 \end{cases}.$$

(b) Let

$$f(x) = \begin{cases} kx + 1 & x \geq k \\ 3x + 1 & x < k \end{cases}.$$

Solution:

(a) Notice that each component of $f(x)$ is continuous so we need only check for continuity at $x = 0$, the point where they “switch”. To do so, we first need to compute the left- and right-handed limits of $f(x)$ at $x = 0$. We see that

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} k^5x - 1 = -1$$

and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} k\sqrt{x+4} + k^2 = 2k + k^2.$$

For continuity, we need both of these limits to be equal to $f(0) = 2k + k^2$. The right-handed limit is always equal to $f(0)$, so we need

$$-1 = \lim_{x \rightarrow 0^-} f(x) = f(0) = 2k + k^2 \implies k^2 + 2k + 1 = (k + 1)^2 = 0.$$

Hence, we must have $\boxed{k = -1}$ for $f(x)$ to be continuous.

For differentiability, note that each component of $f(x)$ is differentiable so we need only check that it is differentiable at $x = 0$, that is, the following limit exists:

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - (2k + k^2)}{h}.$$

Should this limit exist, then we know that $f(x)$ is continuous at $x = 0$ as well which, by the first part of this problem, requires that $k = -1$. Now, for this limit to exist, we need the left- and right-handed limit to agree, so we compute those. Note that

$$\lim_{h \rightarrow 0^-} \frac{f(h) - (2(-1) + (-1)^2)}{h} = \lim_{h \rightarrow 0^-} \frac{(-1)^5h - 1 - (-1)}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h \rightarrow 0^+} \frac{f(h) + 1}{h} = \lim_{h \rightarrow 0^+} \frac{-\sqrt{h+4} + (-1)^2 + 1}{h} = \lim_{h \rightarrow 0^+} \frac{2 - \sqrt{h+4}}{h}$$

which is an indeterminate form. Multiplying the top and bottom by the conjugate, we see that

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{2 - \sqrt{h+4}}{h} &= \lim_{h \rightarrow 0^+} \frac{(2 - \sqrt{h+4})(2 + \sqrt{h+4})}{h(2 + \sqrt{h+4})} \\ &= \lim_{h \rightarrow 0^+} \frac{4 - (h+4)}{h(2 + \sqrt{h+4})} \\ &= \lim_{h \rightarrow 0^+} \frac{-1}{2 + \sqrt{h+4}} = -\frac{1}{4}. \end{aligned}$$

As $-1 \neq -1/4$, we conclude that $\boxed{\text{no such } k \text{ exists.}}$

- (b) Note that each component of $f(x)$ is continuous so we need only check that $f(x)$ is continuous at $x = k$. To do this, we first need to compute the left- and right-handed limit as x approaches k . Note that

$$\lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} 3x + 2 = 3k + 1$$

and

$$\lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} kx + 1 = k^2 + 1.$$

For continuity at $x = k$, we need both of the one-sided limit to be equal to $f(k) = k^2 + 1$. Since the right-handed limit is always equal to $f(k)$, we only need

$$3k + 1 = \lim_{x \rightarrow k^-} f(x) = f(k) = k^2 + 1 \implies \boxed{k = 0, 3.}$$

Since each component of $f(x)$ is differentiable, we only need $f(x)$ to be differentiable at $x = k$, that is,

$$\lim_{h \rightarrow 0} \frac{f(k+h) - f(k)}{h} = \lim_{h \rightarrow 0} \frac{f(k+h) - (k^2 + 1)}{h}$$

must exist. Moreover, if $f(x)$ is to be differentiable at $x = k$ then it must be continuous there so, by the above part, $k = 0, 3$ or (equivalently) $3k - k^2 = 0$. For it to exist, we need the left- and right-handed limits to agree, so we compute these. Note that

$$\lim_{h \rightarrow 0^-} \frac{f(k+h) - (k^2 + 1)}{h} = \frac{3(k+h) + 1 - (k^2 + 1)}{h} = \lim_{h \rightarrow 0^-} \frac{3k - k^2}{h} + \frac{3h}{h} = 3 + \lim_{h \rightarrow 0^-} \frac{3k - k^2}{h} = 3$$

and

$$\lim_{h \rightarrow 0^+} \frac{f(k+h) - (k^2 + 1)}{h} = \lim_{h \rightarrow 0^+} \frac{k(k+h) + 1 - (k^2 + 1)}{h} = \lim_{h \rightarrow 0^+} \frac{kh}{h} = k.$$

Hence, we must have $\boxed{k = 3}$ for $f(x)$ to be differentiable.

Exercise 3. (Limits)

Compute the following limits:

(a)

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$$

(b)

$$\lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2}$$

(c)

$$\lim_{x \rightarrow \infty} \frac{x^2 - 9x + 2}{3x^2 - 7x + 1}$$

(d)

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x}$$

Solution:(a) Note that $x^3 - 1 = (x - 1)(x^2 + x + 1)$ so

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1} x^2 + x + 1 = \boxed{3}.$$

(b) This is an indeterminate form (0/0) with a radical in the numerator so we multiply both the top and bottom by the conjugate. This gives us

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2} &= \lim_{x \rightarrow 2} \frac{(\sqrt{x+2} - 2)(\sqrt{x+2} + 2)}{(x - 2)(\sqrt{x+2} + 2)} \\ &= \lim_{x \rightarrow 2} \frac{x + 2 - 2^2}{(x - 2)(\sqrt{x+2} + 2)} \\ &= \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x+2} + 2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x+2} + 2} \\ &= \boxed{\frac{1}{4}}. \end{aligned}$$

(c) As we let x tend towards infinity, this limit will depend only on the largest powers of the numerator and denominator, that is,

$$\lim_{x \rightarrow \infty} \frac{x^2 - 9x + 2}{3x^2 - 7x + 1} = \lim_{x \rightarrow \infty} \frac{x^2}{3x^2} = \boxed{\frac{1}{3}}.$$

(d) Note that our limit takes the indeterminate form ∞/∞ so we may apply L'Hopital's rule to get

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} \\ &= \lim_{x \rightarrow \infty} \frac{1/x}{1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= \boxed{0}. \end{aligned}$$

Exercise 4. (Derivatives)

Compute the derivative of:

(a)

$$f(x) = \frac{\tan(e^x)}{2x^2 + 1}.$$

(b)

$$f(x) = (\sin 2x)^{\sqrt{3x}}.$$

Solution:

(a) Note that we can write

$$f(x) = \frac{g(x)}{h(x)}$$

where $g(x) = \tan(e^x)$ and $h(x) = 2x^2 + 1$. So, we want to apply the quotient rule which requires us to compute $g'(x)$ and $h'(x)$. To compute $g'(x)$, we see that we have an “inside” function so we need to apply the chain rule:

$$g'(x) = (\sec(e^x))^2 \frac{d}{dx} e^x = (\sec(e^x))^2 e^x.$$

To compute $h'(x)$, we only need to apply linearity and the power rule:

$$h'(x) = 2(2x) = 4x.$$

Now, we use the quotient rule to compute $f'(x)$:

$$f'(x) = \frac{(2x^2 + 1) ((\sec(e^x))^2 e^x) - (\tan(e^x)) (4x)}{(2x^2 + 1)^2}.$$

(b) First, note that we may write

$$f(x) = (\sin 2x)^{\sqrt{3x}} = e^{\sqrt{3x} \ln(\sin 2x)}.$$

So, we'll need to apply the chain rule to the exponent when computing the derivative of $f(x)$. We see that

$$\frac{d}{dx} [\sqrt{3x} \ln(\sin 2x)] = \frac{3 \ln(\sin 2x)}{2\sqrt{3x}} + \sqrt{3x} \left(\frac{2 \cos 2x}{\sin 2x} \right)$$

being sure to apply the chain rule. Note we see that

$$f'(x) = e^{\sqrt{3x} \ln(\sin 2x)} \frac{d}{dx} [\sqrt{3x} \ln(\sin 2x)] = (\sin 2x)^{\sqrt{3x}} \left(\frac{3 \ln(\sin 2x)}{2\sqrt{3x}} + \sqrt{3x} \left(\frac{2 \cos 2x}{\sin 2x} \right) \right).$$

Exercise 5. (Rate of change)

A particle moves along a line with position $s(t) = \frac{1}{3}t^3 - 2t^2 + 3$ (meters) at time t (seconds).

- (a) Find the velocity function.
- (b) At what time t is the particle at rest?
- (c) On which time intervals, is the particle speeding up? Speeding down?

Solution:

- (a) We know that the velocity function, $v(t)$, is the derivative of $s(t)$ so

$$v(t) = s'(t) = \boxed{t^2 - 4t}.$$

- (b) The particle is at rest when $v(t) = 0$. By part (a), this occurs when

$$0 = v(t) = t^2 - 4t = t(t - 4) \implies \boxed{t = 0, 4}$$

are all times when the particle is at rest.

- (c) To deduce when the particle is slowing down or speeding up, we need to know the acceleration of our particle, that is, the derivative of our velocity:

$$a(t) = v'(t) = 2t - 4.$$

Now, our particle is slowing down on intervals when the sign of $v(t)$ and $a(t)$ differ and speeding up on intervals when the sign of $v(t)$ and $a(t)$ agree. The only possible points when $v(t)$ or $a(t)$ can switch signs is when they're zero. So, $a(t)$ has constant sign on the interval $[0, 2)$ and $(2, \infty)$ and $v(t)$ has constant sign on the interval $(0, 4)$ and $(4, \infty)$. Hence, we only need to check the sign of each on these intervals which can be done by evaluating at some point on this interval. Note

$$\begin{array}{ll} a(0) = -4 & < 0 \\ a(3) = 2 & > 0 \\ v(1) = -3 & < 0 \\ v(5) = 5 & > 0 \end{array}$$

which we depict pictorially by:

$$\begin{array}{l} \text{sign}(v(t)):\quad \begin{array}{c} - \qquad \qquad \qquad + \\ \overbrace{\hspace{1.5cm}} \quad \overbrace{\hspace{1.5cm}} \\ 0 \qquad \qquad 2 \qquad \qquad 4 \end{array} \\ \text{sign}(a(t)):\quad \begin{array}{c} - \qquad \qquad \qquad + \\ \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{1.5cm}} \end{array} \end{array}$$

Hence, by comparing signs, we know the particle is

$$\boxed{\text{speeding up on } (0, 2) \cup (4, \infty)}$$

and

$$\boxed{\text{slowing down on } (2, 4)}.$$

Exercise 6. (Tangent lines)

Let

$$f(x) = e^{2x} \sin 2x, \quad 0 \leq x \leq \pi.$$

Find all $x \in [0, \pi]$ on the graph of $f(x)$ where the tangent line has slope 0 and the equation of these tangent lines.

Solution: The slope of a tangent line on the graph of $f(x)$ at $x = a$ is given by $f'(a)$, so we need to compute $f'(x)$. To do so, we use the product rule and chain rule to get:

$$f'(x) = 2e^{2x} \sin 2x + 2e^{2x} \cos 2x.$$

As we want points where the slope of the tangent line has slope zero, we need

$$0 = f'(x) = 2e^{2x} \sin 2x + 2e^{2x} \cos 2x = 2e^{2x}(\sin 2x + \cos 2x) \implies \sin 2x = -\cos 2x$$

as $2e^{2x}$ is never zero. The only values of sin and cos which differ by a sign (with inputs between 0 and 2π) are $3\pi/4$ and $7\pi/4$ so

$$2x = \frac{3\pi}{4}, \frac{7\pi}{4} \implies \boxed{x = \frac{3\pi}{8}, \frac{7\pi}{8}}.$$

The corresponding tangent lines will take the form $y - f(x_0) = f'(x_0)(x - x_0)$ where x_0 is $3\pi/8$ or $7\pi/8$. We know that the slope at these points is zero so the equations for our tangent lines become

$$y - f\left(\frac{3\pi}{8}\right) = \boxed{y - \frac{\sqrt{2}e^{3\pi/4}}{2} = 0} \quad \text{and} \quad y - f\left(\frac{7\pi}{8}\right) = \boxed{y + \frac{\sqrt{2}e^{7\pi/4}}{2} = 0}.$$

Exercise 7. (Implicit differentiation)

Compute $\frac{dy}{dx}$ for

$$3xy - xy^3 - 2 = 0.$$

Find all points on the graph of the above equation whose tangent line has an infinite slope. Find the tangent line of the above equation at the point $(1, 2)$.

Solution: Recall that we consider y to be the dependent variable, that is, a function of x so the chain rule applies. Note that

$$\begin{aligned} 0 &= \frac{d}{dx}(3xy - xy^3 - 2) \\ &= 3\frac{d}{dx}(xy) - \frac{d}{dx}(xy^3) \\ &= 3\left(y + x\frac{dy}{dx}\right) - \left(y^3 + 3xy^2\frac{dy}{dx}\right). \end{aligned}$$

Solving for $\frac{dy}{dx}$, we find that

$$\boxed{\frac{dy}{dx} = \frac{y^3 - 3y}{3x - 3xy^2}}.$$

Note that our tangent line has infinite slope only when the denominator of dy/dx is zero, that is,

$$3x - 3xy^2 = 3x(1 - y^2) = 0.$$

In our original equation, we cannot have $x = 0$ (as $-2 \neq 0$) so we know that $3x \neq 0$ which implies we must have $1 - y^2 = 0$ so $y = \pm 1$. We now know the y -coordinate of these points, all that remains to determine are the x -coordinate of these points. To do so, we'll plug the value we found for y into our original equation and solve for x :

$$0 = 3x(\pm 1) - x(\pm 1)^3 - 2 = \pm(3x - x) - 2 = \pm 2x - 2 \Rightarrow x = \pm 1.$$

Thus, the points on our equation at which the tangent line has infinite slope are exactly

$$\boxed{(1, 1)} \quad \text{and} \quad \boxed{(-1, -1)}.$$

The tangent line at a given point (x_0, y_0) is just dy/dx evaluated at $x = x_0$ and $y = y_0$. In our case, $x_0 = 1$ and $y_0 = 2$ so dy/dx becomes

$$\frac{2^3 - 3(2)}{3(1) - 3(1)(2^2)} = \boxed{-\frac{2}{9}}.$$

Exercise 8. (Derivatives of inverse functions)

Compute $(f^{-1})'(a)$ for the given $f(x)$ and a :

(a) $f(x) = x^3 + 2x - 3$ with $a = 0$.

(b) $f(x) = x - \frac{2}{x}$, for $x > 0$, with $a = 1$.

Solution: Recall that if $f(b) = a$ and $f'(b) \neq 0$ then

$$(f^{-1})'(a) = \frac{1}{f'(b)}.$$

We'll be making use of this theorem for this problem.

- (a) To use the theorem initially stated, we need to find such a b and $f'(x)$. We know how to compute $f'(x)$ using the power rule and linearity:

$$f'(x) = 3x^2 + 2.$$

To find b , we need $f(b) = a$ so, in our case,

$$b^3 + 2b - 3 = 0.$$

That is, b is a root of $f(x)$. Since f is a polynomial with integer coefficients and a leading term of 1, the rational root theorem tells us that any root which is an integer must divide 3. Hence, we need only check $\pm 1, \pm 3$ to exclude the possibility of a root in the integers. Note that

$$f(1) = 1^3 + 2(1) - 3 = 0$$

so $b = 1$ works. Thus, we have that

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \boxed{\frac{1}{5}}.$$

- (b) Just as in part (a), we need to find $f'(x)$ and b such that $f(b) = 1$, with the additional restriction that $b > 0$. Once again, using the power rule and linearity to compute $f'(x)$, we see that:

$$f'(x) = 1 + \frac{2}{x^2}.$$

Now, we want to find $b > 0$, that is, we want

$$b - \frac{2}{b} = 1 \implies b^2 - 2 = b \implies b^2 - b + 2 = (b - 1)(b + 2) = 0.$$

Since we want $b > 0$, we take our b to be 1 (rather than -2). Now, using the theorem state initially, we see that

$$(f^{-1})'(1) = \frac{1}{f'(1)} = \boxed{\frac{1}{3}}.$$

Exercise 9. (Linear approximation)

Use linear approximations to approximate the following values with the given function and point, a .

(a) $\sqrt{16 + 1/100}$ with $f(x) = \sqrt{x}$ and $a = 16$.

(b) $\sin\left(\frac{314}{100}\right)$ with $f(x) = \sin x$ and $a = \pi$.

Solution:

(a) Using the power rule, we know that $f'(x) = \frac{1}{2\sqrt{x}}$ so

$$f'(a) = f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}.$$

Moreover, we know that $f(16) = 4$ so the linear approximation of $f(x)$ at $x = 16$ is

$$L(x) = f(16) + f'(16)(x - 16) = 4 + \frac{1}{8}(x - 16).$$

Now, we see that

$$\sqrt{16 + 1/100} = f(16 + 1/100) \approx L(16 + 1/100) = \boxed{4 + \frac{1}{8} \left(\frac{1}{100} \right)}.$$

(b) We know that $f'(x) = \cos x$ so

$$f'(a) = f'(\pi) = -1.$$

Moreover, we know that $f(\pi) = 0$ so the linear approximation of $f(x)$ at $x = \pi$ is

$$L(x) = f(\pi) + f'(\pi)(x - \pi) = -(x - \pi) = \pi - x.$$

Now, we see that

$$\sin\left(\frac{314}{100}\right) = f\left(\frac{314}{100}\right) \approx L\left(\frac{314}{100}\right) = \boxed{\pi - \frac{314}{100}}.$$

Exercise 10. (Minima/Maxima)

Find the minima and maxima of the given functions over the given intervals:

- (a) $f(x) = x \sin x + \cos x$ where $x \in [0, 2\pi]$.
 (b) $f(x) = x^x$ where $x \in (0, \infty)$.

Solution:

- (a) First, we compute the critical points of $f(x)$. To do so, we first compute $f'(x)$:

$$f'(x) = \sin x + x \cos x - \sin x = x \cos x.$$

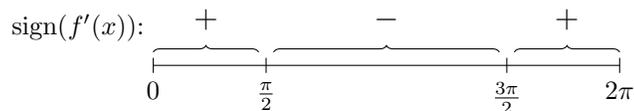
Next, we set $f'(x)$ equal to 0 and solve for x :

$$0 = f'(x) = x \cos x \implies x = 0, \frac{\pi}{2}, \frac{3\pi}{2}.$$

Now, we employ the first derivative test. As the sign of $f'(x)$ can only change at critical points, we need only check the sign of $f'(x)$ on $(0, \pi/2)$, $(\pi/2, 3\pi/2)$, and $(3\pi/2, 2\pi)$ which can be done by evaluation:

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= \frac{\pi}{4} \left(\frac{\sqrt{2}}{2}\right) &> 0 \\ f'(\pi) &= \pi(-1) &< 0 \\ f'\left(\frac{7\pi}{4}\right) &= \frac{7\pi}{4} \left(\frac{\sqrt{2}}{2}\right) &> 0 \end{aligned}$$

We can depict this pictorially as



Hence, we have

$$\boxed{\text{minima of: } f(0) = 1, \text{ and } f(3\pi/2) = -3\pi/2}$$

and

$$\boxed{\text{maxima of: } f(\pi/2) = \pi/2, \text{ and } f(2\pi) = 1}$$

- (b) Since the interval that we're maximizing/minimizing over does not include the endpoints, we need only find the critical points and determine whether they are minima, maxima, or neither. Write $f(x) = x^x = e^{x \ln x}$ then, using the chain rule and product rule, we see that

$$f'(x) = e^{x \ln x} \left(\ln x + x \left(\frac{1}{x} \right) \right) = x^x (\ln x + 1).$$

Since $x^x \neq 0$ when $x \in (0, \infty)$, we see that

$$0 = f'(x) = x^x (\ln x + 1) \implies \ln x + 1 = 0 \implies x = \frac{1}{e}.$$

Now, we use the first derivative test to determine whether this is a maximum, minimum, or neither. Note the $x^x > 0$ for $x \in (0, \infty)$ so the sign of $f'(x) = x^x(\ln x + 1)$ depends only on the sign of $\ln x + 1$. Note that

$$\ln\left(\frac{1}{e^2}\right) + 1 = -2 + 1 = -1 < 0 \quad \text{and} \quad \ln(1) + 1 = 0 + 1 = 1 > 0$$

so, pictorially, we have:

$$\text{sign}(f'(x)): \quad \begin{array}{c} - \qquad \qquad \qquad + \\ \underbrace{\hspace{1.5cm}} \quad \underbrace{\hspace{2.5cm}} \\ \qquad \qquad \qquad | \qquad \qquad \qquad \rightarrow \\ \qquad \qquad \qquad 1/e \end{array}$$

Hence, we conclude that we have

a minimum of: $f(1/e) = \left(\frac{1}{e}\right)^{1/e}$.

Exercise 11. (Mean Value Theorem)

- (a) Use the mean value theorem to show that $e^x - e$ has exactly one root.
(b) Determine whether the mean value theorem applies to

$$f(x) = \ln(2x - 1), \quad 1 \leq x \leq 3.$$

If the mean value theorem applies, find all $c \in (1, 3)$ such that

$$(3 - 1)f'(c) = f(3) - f(1).$$

Solution:

- (a) Let $f(x) = e^x - e$ and note that $f(1) = e - e = 0$ so $f(x)$ has at least one root. Let's try applying the mean values theorem to $f(x)$ on the interval $[1, b]$ for some real number $b \neq 1$. We can do so as $f(x)$ is differentiable everywhere, so it is also continuous everywhere. In particular, it is continuous on $[1, b]$ and differentiable on $(1, b)$ so the mean value theorem applies. The mean value theorem tells us that there exists some $c \in (1, b)$ such that

$$f'(c) = \frac{f(b) - f(1)}{b - 1}.$$

We know that $f(1) = 0$ and we know that $f'(x) = e^x$, so the above becomes

$$e^c = \frac{f(b)}{b - 1}.$$

Now, suppose that b is another root of $f(x)$ which is distinct from 1 then the above becomes

$$e^c = \frac{0}{b - 1} = 0.$$

But, we know that e^x is always positive so this cannot happen. Thus, we deduced that $f(x)$ has exactly one root using only the mean value theorem.

- (b) For the mean value theorem to apply, we need $f(x)$ to be continuous on $[1, 3]$ and differentiable on $(1, 3)$. We know that $\ln x$ is differentiable and continuous on $(0, \infty)$ so $f(x)$ is as well. In particular, $f(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$ so the mean value theorem applies to $f(x)$ on $[1, 3]$.

Now, we want $c \in (1, 3)$ such that

$$(3 - 1)f'(c) = f(3) - f(1).$$

(*What guarantees that such a c exists?*) First, we compute $f'(x)$, remembering to apply the chain rule:

$$f'(x) = \frac{2}{2x - 1}.$$

So, we want $c \in (1, 3)$ satisfying

$$2 \left(\frac{2}{2c - 1} \right) = \ln(6 - 1) - \ln(2 - 1) = \ln 5 \implies \frac{4}{\ln 5} = 2c - 1 \implies c = \frac{1}{2} \left(\frac{4}{\ln 5} + 1 \right).$$