A Combinatorial Proof of Schläfli's and Gould's Identity

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Outline

Stirling Permutations

Definition [GS78]

Stirling permutations are permutations of the multiset $\{1, 1, 2, 2, \ldots, k, k\}$ which avoid 212, i.e. if π is a Stirling permutation then there is no subsequence of the form aba where $b < a$. Denote the set of all Stirling permutations as $\mathcal{Q}_k$. The number of **descents** in π, $\operatorname{des}\pi$, is the number of i such that $\pi_i > \pi_{i+1}$ or $i = 2k$.

Ex: $\pi = 1245543321 \in \mathcal{Q}_5$ **Non-Ex:** $\sigma = 1245533421 \notin \mathcal{Q}_5$ **Ex:**

$\pi = 1245543321 \in \mathcal{Q}_5$ **Non-Ex:** $\sigma = 1245533421 \notin \mathcal{Q}_5$

$\operatorname{des}\pi = 5$ $\operatorname{des}\sigma = 4$

Stirling Permutations

Stirling Numbers of the 2nd Kind

Defintion

The **Stirling Numbers of the 2nd Kind** (OEIS A008277) is the sequence defined by

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

with $S(0,0) = 1$ and $S(n,k) = 0$ if $n,k < 0$ or $n > k$. Gessel and Stanley proved that

$$S(k+n,n) = \sum_{\pi \in \mathcal{Q}_k} \binom{2k+n-\operatorname{des}\pi}{2k}.$$

Stirling Numbers

Stirling Numbers of the 1st Kind

Definition

The **unsigned Stirling Numbers of the 1st kind** (OEIS A008275) is the sequence defined by

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

with the same initial conditions. Set $s(n,k) = (-1)^{n-k}c(n,k)$. Gessel and Stanley proved that

$$s(n,n-k) = (-1)^k \sum_{\pi \in \mathcal{Q}_k} \binom{n + \operatorname{des}\pi - 1}{2k}.$$

Schläfli's and Gould's Identity

Theorem [Sch67, Gou60]

Let $0 \leq n, k$ then

$$s(n, n-k) = \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} S(k+\alpha, \alpha)$$

and

$$S(n, n-k) = \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} s(k+\alpha, \alpha).$$

An Observation

Note

$$\sum_{\pi\in\mathcal{Q}_k}(-1)^k\binom{n+\operatorname{des}\pi-1}{2k}=s(n,n-k)$$

$$=\sum_{\alpha=0}^{k}(-1)^{\alpha}\binom{n+k}{k-\alpha}\binom{n+\alpha-1}{k+\alpha}S(k+\alpha,\alpha)$$

$$=\sum_{\pi\in\mathcal{Q}_k}\sum_{\alpha=\operatorname{des}\pi}^{k}(-1)^{\alpha}\binom{n+k}{k-\alpha}\binom{n+\alpha-1}{k+\alpha}\binom{2k+\alpha-\operatorname{des}\pi}{2k}.$$

The Identity

Theorem

For $0 \leq n, k, i$ and $j \in \mathbb{Z}$ with $i \leq k$,

$$\sum_{\alpha=i}^{k} (-1)^{\alpha} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{k+j+\alpha} \binom{2k-i+j+\alpha}{2k+j} = (-1)^{k} \binom{n+i-1}{2k+j}.$$

Schläfli's and Gould's Identity Revisited

Theorem

For $0 \leq n, k, \gamma$,

$$\sum_{\alpha=0}^{k+\gamma}(-1)^{\alpha+\gamma}\binom{n+k+\gamma}{k+\gamma-\alpha}\binom{n+\alpha-1}{k-\gamma+\alpha}S(k+\alpha,\alpha)=s(n,n-k)$$

and

$$\sum_{\alpha=0}^{k+\gamma}(-1)^{\alpha+\gamma}\binom{n+k+\gamma}{k+\gamma-\alpha}\binom{n+\alpha-1}{k-\gamma+\alpha}s(k+\alpha,\alpha)=S(n,n-k).$$

General Sequences

Theorem

Let $k \geq 1$ and $A \in \mathbb{C}[t]$ with $0 \leq \deg A \leq k$. If

$$\sum_{n \geq 0} f(n) t^n = \frac{tA(t)}{(1-t)^{k+1}}$$

then

$$f(n) = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{\alpha} g(\alpha)$$

where

$$\sum_{n \geq 0} g(n) t^n = \frac{t^k A(1/t)}{(1-t)^{k+1}}.$$

An Application

Eulerian Polynomial

Set $A_k(t) = \sum_{\sigma \in S_k} t^{\operatorname{des}\sigma}$, then it is well-known that

$$\sum_{n \geq 0} n^k t^n = \frac{A_k(t)}{(1-t)^{k+1}}.$$

Moreover, due to the map $\sigma_1 \dots \sigma_k \mapsto \sigma_k \dots \sigma_1$, we get that $A_k(t) = t^{k+1} A_k(1/t)$. Thus, we get

$$n^k = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{\alpha} \alpha^k$$

which is known [Chu23].

An Application

Lah Numbers

The **Lah Numbers** (OEIS A271703), denoted $L(n,k)$, are

$$\frac{n!}{k!}\binom{n-1}{k-1}$$

and have the expontential generating function (for fixed $k \geq 1$)

$$\sum_{n\geq 0} \frac{L(n+k,k)}{(n+k)!} t^{n+1} = \frac{t/k!}{(1-t)^k}.$$

So, we have

$$\frac{L(n+k,k)}{(n+k)!} = \sum_{\alpha=1}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{\alpha-1} \frac{L(\alpha,k)}{\alpha!}.$$

An Application

Narayana Numbers

The k-th **Narayana polynomial** is defined as

$$N_k(t) = \sum_{D \in \mathcal{D}_k} t^{\mathrm{peak}(D)} = \sum_{n \geq 0} N_{k,n} t^n$$

where $\mathcal{D}_k$ is the set of Dyck paths of semilength k. Then, for fixed $k \geq 1$,

$$\frac{N_k(t)}{(1-t)^{2k+1}} = \sum_{n \geq 0} N_{k+n,n} t^n.$$

So, we have

$$N_{k+n,n} = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+2k}{n+k+\alpha} \binom{n+k+\alpha-1}{k+\alpha} N_{k+\alpha,\alpha}.$$

An Application

Quasi-Stirling Permutations

Definition

A **Quasi-Stirling permutation** is a permutation of the multiset $\{1, 1, 2, 2, \ldots k, k\}$ which avoids 1212 and 2121, i.e. no subsequences of the form *abab*. Denote the set of all Quasi-Stirling permutations by $\overline{\mathcal{Q}}_k$. Elizalde [Eli20] proved that

$$\sum_{\pi \in \overline{\mathcal{Q}}_k} \binom{2k + n - \operatorname{des} \pi}{2k} = \frac{n^k}{k+1} \binom{n+k}{n}.$$

A bijective proof was provided by Yan and Zhu [YZ21].

An Application

Quasi-Stirling Permutations

Using generating functions, we showed that

$$\sum_{\pi \in \overline{\mathcal{Q}}_k} \binom{n + \operatorname{des} \pi - 1}{2k} = \binom{n}{k+1} n^{k-1}.$$

So, we have that

$$\frac{1}{k+1} \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{k+\alpha} \binom{k+\alpha}{\alpha} \alpha^k = \binom{n}{k+1} n^{k-1}.$$

Thank You for Listening!
Questions?

`am3l14.github.io`

References

[Chu23] Wenchang Chu. Reciprocal Symmetry via Inverse Series Pairs. *Symmetry*, 15(5):1086, May 2023.

[Eli20] Sergi Elizalde. Descents on quasi-Stirling permutations, February 2020. arXiv:2002.00985 [math].

[Gou60] H. W. Gould. Stirling number representation problems. *Proceedings of the American Mathematical Society*, 11(3):447–451, 1960.

[GS78] Ira Gessel and Richard P Stanley. Stirling polynomials. *Journal of Combinatorial Theory, Series A*, 24(1):24–33, January 1978.

[Sch67] Ludwig Schläfli. Ergänzung der abhandlung über die entwickelung des products $1.(1+x)(1+2x)(1+3x)\ldots(1+(n-1)x) = \prod^{n}(x)$. *Journal für die reine und angewandte Mathematik*, 67:179–182, 1867.

[YZ21] Sherry H. F. Yan and Xue Zhu. Quasi-Stirling