Combinatorial Interpretation of Rascal Numbers

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Outline

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1. The Basics
2. Revisiting the Original Recurrence

Outline

Rascal Triangle

Original Recurrence

Rascal Triangle

Original Recurrence

Consider the following:

$$\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{array}$$

Rascal Triangle

Original Recurrence

Consider the following:

$$
\begin{array}{lllll}
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\end{array}
$$

Rascal Triangle

Original Recurrence

Consider the following:

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$$

Anggorro et al. defined the rascal triangle (A077028) by

$$R_{n,k} = \frac{R_{n-1,k}R_{n-1,k-1}+1}{R_{n-2,k-1}}$$

with $R_{0,n} = R_{n,n} = 1$ for $n \geq 0$ and $R_{n,k} = 0$ if $n, k < 0$ or $n < k$.

Rascal Triangle

Original Recurrence

Consider the following:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	5	4	1		
1	5	7	7	5	1	
1	6	9	10	9	6	1

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Rascal Triangle

Fleron's Reccurence

Rascal Triangle

Fleron's Reccurence

Fleron showed that

$$R_{n,k} = R_{n-1,k} + R_{n-1,k-1} - R_{n-2,k-1} + 1$$

with the same initial conditions.

Combinatorial Interpretation

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We noticed that

$$\sum_{k=0}^{n} R_{n,k} = \binom{n+1}{3} + n + 1.$$

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Theorem

$R_{n,k}$ is the number of binary words of length n with k 1s that have at most 1 ascent.

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$$\sum_{k=0}^{n} R_{n,k} = \binom{n+1}{3} + n + 1.$$

Theorem

$R_{n,k}$ is the number of binary words of length n with k 1s that have at most 1 ascent. Denote the set of all such words as $B_k(n)$ and set $b_k(n) = |B_k(n)|$.

Combinatorial Interpretation

Proof

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Theorem (Fleron's Recurrence)

For $0 \leq k \leq n$,

$$b_k(n) = b_k(n-1) + b_{k-1}(n-1) - b_{k-1}(n-2) + 1$$

and satisfies the same initial conditions as $R_{n,k}$.

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Case 2. If $w \in B_k(n)$ ends in a 1, remove it to get a word in $B_{k-1}(n-1)$. This word cannot end in a 0 and have 1 ascent. So $b_{k-1}(n-1) - (b_{k-1}(n-2) - 1)$ such w. □

Revisiting the Original Recurrence

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For $0 < k < n$,

$$R_{n,k} = \frac{R_{n-1,k}R_{n-1,k-1} + 1}{R_{n-2,k-1}}.$$

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Proof (Sketch).

We will equivalently show that

$$R_{n,k}R_{n-2,k-1} - R_{n-1,k}R_{n-1,k-1} = 1.$$

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$R_{n-1,k}$ counts words in $B_k(n)$ ending in at least one 0.

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$R_{n-2,k-1}$ counts words in $B_k(n)$ beginning in at least one 1 and ending in at least one 0. □

Revisiting the Original Recurrence

Proof Continued

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$R_{n,k}R_{n-2,k-1}$ counts $S = \left\{(w, 1\ z\ 0) \in (B_k(n))^2\right\}$.

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Pair off elements in $T \cap S$ with the inclusion map.

Revisiting the Original Recurrence

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Pair off elements in $T \cap S$ with the inclusion map. Apply $(w, z) \mapsto (z, w)$ to "appropriate" elements of T.

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Pair off elements in $T \cap S$ with the inclusion map. Apply $(w, z) \mapsto (z, w)$ to "appropriate" elements of T. Apply

$$(0^y 1^k 0^{n-k-y},\ 1^x 0^{n-k} 1^{k-x}) \mapsto (0^{n-k} 1^k,\ 1^x 0^y 1^{k-x} 0^{n-k-y})$$

to all remaining elements of T. $\square$

Revisiting the Original Recurrence

Proof Concluded

Theorem

For $0 < k < n$,

$$R_{n,k} = \frac{R_{n-1,k}R_{n-1,k-1} + 1}{R_{n-2,k-1}}.$$

Proof (Sketch).

Revisiting the Original Recurrence

Proof Concluded

Theorem

For $0 < k < n$*,*

$$R_{n,k} = \frac{R_{n-1,k}R_{n-1,k-1} + 1}{R_{n-2,k-1}}.$$

Proof (Sketch).

The only remaining element of S is $(0^{n-k}1^k,\ 1^k0^{n-k})$. $\square$

Generalization

Definition

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When $j = 2$, the first few rows of this sequence are

$$\begin{array}{ccccccc} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 6 & 15 & 19 & 15 & 6 & 1 \end{array}$$

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\end{array}
$$

Generalization

Linear Recurrence

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Theorem

For $0 \le k \le n$ *and* $0 \le j$,

$$R^{(j)}_{n,k} = R^{(j)}_{n-1,k} + R^{(j)}_{n-1,k-1} - R^{(j)}_{n-2,k-1} + R^{(j-1)}_{n-2,k-1}$$

with $R^{(j)}_{n,k} = 0$ *when* $n, k, j < 0$ *or* $n < k$ *and* $R^{(j)}_{0,0} = R^{(j)}_{1,0} = R^{(j)}_{1,1} = 1$.

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Proof.

Case 1. If $w \in B^j_k(n)$ ends in a 0, remove it to get a word in $B^j_k(n-1)$.

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For $0 \le k \le n$ and $0 \le j$,

$$R^{(j)}_{n,k} = R^{(j)}_{n-1,k} + R^{(j)}_{n-1,k-1} - R^{(j)}_{n-2,k-1} + R^{(j-1)}_{n-2,k-1}$$

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Case 1. If $w \in B^j_k(n)$ ends in a 0, remove it to get a word in $B^j_k(n-1)$. So $R^{(j)}_{n-1,k}$ such w.

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Case 2. If $w \in B^j_k(n)$ ends in a 1,

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Theorem

For $0 \le k \le n$ *and* $0 \le j$,

$$R_{n,k}^{(j)} = R_{n-1,k}^{(j)} + R_{n-1,k-1}^{(j)} - R_{n-2,k-1}^{(j)} + R_{n-2,k-1}^{(j-1)}$$

with $R_{n,k}^{(j)} = 0$ *when* $n, k, j < 0$ *or* $n < k$ *and* $R_{0,0}^{(j)} = R_{1,0}^{(j)} = R_{1,1}^{(j)} = 1$.

Proof.

Case 1. If $w \in B_k^j(n)$ ends in a 0, remove it to get a word in $B_k^j(n-1)$. So $R_{n-1,k}^{(j)}$ such w.

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Case 2. If $w \in B^j_k(n)$ ends in a 1, remove it to get a word in $B^j_{k-1}(n-1)$. This word cannot end in a 0 and have j ascents.

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For $0 \leq k \leq n$ and $0 \leq j$,

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Generalization

Equivalent Definition

Generalization

Equivalent Definition

Definition

Gregory et al. defined the set of *rascal subsets* as

$$\binom{[n]}{k}_j = \{S \subseteq [n] : |S \cap [n-k]| \leq j,\ |S| = k\}.$$

A Conjecture of Gregory

A Conjecture of Gregory

Conjecture (Gregory et al., 2023)

For $j \geq 0$,

$$\sum_{k=0}^{4j+3} R^{(j)}_{4j+3,k} = 2^{4j+2}$$

A Conjecture of Gregory

Generalized Row Sum

A Conjecture of Gregory

Generalized Row Sum

Theorem

For $n, j \geq 0$,

$$\sum_{k=0}^{n} R_{n,k}^{(j)} = \sum_{k=0}^{2j+1} \binom{n}{k}.$$

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For $n, j \geq 0$,

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Proof.

Let $B^j(n) = \bigcup_k B_k^j(n)$.

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Let $B^j(n) = \bigcup_k B_k^j(n)$. For $w = w_1 \dots w_n \in B^j(n)$, we note that $\operatorname{asc}(w) \leq j$

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If $w_1 = 0$ then $\mathrm{des}(w) \leq j$,

A Conjecture of Gregory

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Let $B^j(n) = \bigcup_k B_k^j(n)$. For $w = w_1 \ldots w_n \in B^j(n)$, we note that $\mathrm{asc}(w) \leq j$ and $\mathrm{des}(w) \leq j+1$.
If $w_1 = 0$ then $\mathrm{des}(w) \leq j$, so $w \mapsto \mathrm{Des}(w) \cup \mathrm{Asc}(w) \cup \{n\}$.

A Conjecture of Gregory

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For $n, j \geq 0$,

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If $w_1 = 0$ then $\mathrm{des}(w) \leq j$, so $w \mapsto \mathrm{Des}(w) \cup \mathrm{Asc}(w) \cup \{n\}$.

If $w_1 = 1$ then $w \mapsto \mathrm{Des}(w) \cup \mathrm{Asc}(w)$. $\square$

A Conjecture For You

Generalization of Original Recurrence

A Conjecture For You

Generalization of Original Recurrence

Conjecture

For $0 < k < n$ and $0 \leq j$,

$$R_{n,k}^{(j)} = \frac{R_{n-1,k}^{(j)} R_{n-1,k-1}^{(j)} + E(n,k,j)}{R_{n-2,k-1}^{(j)}}.$$

We conjecture that E has a "nice" closed form.

A Conjecture For You

Generalization of Original Recurrence

Conjecture

For $0 < k < n$ and $0 \leq j$,

$$R^{(j)}_{n,k} = \frac{R^{(j)}_{n-1,k} R^{(j)}_{n-1,k-1} + E(n,k,j)}{R^{(j)}_{n-2,k-1}}.$$

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We know that $E(n,k,0) = 0$ and $E(n,k,1) = 1$ works.

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Through numerical tests, we’ve found $E(n,k,2) = R^{(2)}_{2-n,1-k}$ which we’ve confirmed algebraically.

The End

Thanks for listening!

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Questions?

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