Calibration of the method of images for regularized Stokeslets using sphere motion near a boundary

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Abstract

Many numerical simulations in fluid dynamics require modeling a sphere in motion near a boundary. In Stokes flow, the method of images for regularized Stokeslets (MIRS) has been widely used and validated with theoretical results for the rotational and translational motions of spheres parallel or perpendicular to a boundary, respectively. Our work, taking into account all possible motions of a unit sphere, presents a systematic study that calibrates the MIRS with the theory and dynamically similar experiments.

We discover that the surface discretization called spherical centroidal Voronoi tessellations (SCVT) is the most accurate and robust for all motions as we compare SCVT with those discretizations whose point distributions on the sphere's surface are symmetric with respect to the boundary. Depending on which regularization function used in the MIRS, we find a constant ratio, for all motions, of the optimal regularization parameter in free space to the average distance used in the SCVT discretization. Our study reveals how the discretization type and size, optimal regularization parameter, and regularization function affect the accuracy and robustness of sphere-motion simulations.

Method of Regularized Stokeslets (MRS) [1]

The velocity $\boldsymbol{u}$ and pressure p of the fluid are found by solving the incompressible Stokes equations in three dimensions:

$$\mu\Delta\boldsymbol{u} = \nabla p - \boldsymbol{F} \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad (2)$$

where μ is the fluid viscosity. $\boldsymbol{F}$ is the force density represented as $\boldsymbol{f_0}\phi_\epsilon(\widehat{\boldsymbol{x}}-\boldsymbol{x_0})$ or $\boldsymbol{f_0}\psi_\epsilon(\widehat{\boldsymbol{x}}-\boldsymbol{x_0})$ where $\boldsymbol{f_0}$ is a point force at a point $\boldsymbol{x_0}$ and $\phi_\epsilon, \psi_\epsilon$ are the following regularization functions:

$$\phi_\epsilon(r) = \frac{15\epsilon^4}{8\pi(r^2+\epsilon^2)^{7/2}},\ \psi_\epsilon(r) = \frac{15\epsilon^4(40\epsilon^6-132\epsilon^4r^2+57\epsilon^2r^4-2r^6)}{16\pi(r^2+\epsilon^2)^{13/2}}$$

where $r = |\widehat{\boldsymbol{x}}-\boldsymbol{x_0}|$.

[1] R. Cortez, L. Fauci, A. Medovikov. The method of regularized Stokeslets in three dimensions: Analysis, validation, and application to helical swimming, *Physics of Fluids*, vol. 17, no. 3, 1–14, 2005

Method of Images for Regularized Stokeslets (MIRS) [2]

Suppose there is a boundary at $x = w$ and a Stokeslet of strength $\boldsymbol{f_0} = (f_1, f_2, f_3)$ centered at $\boldsymbol{x_0} = (w+h, y, z)$. Set $\boldsymbol{x_{0,im}} = (w-h, y, z)$ and let $\boldsymbol{x_e}$ be a point on the boundary. To cancel the fluid velocity at $\boldsymbol{x_e}$ we impose a Stokeslet of strength $-\boldsymbol{f_0}$, a potential dipole of strength $h^2(-f_1, f_2, f_3)$, a Stokeslet doublet of strength $2h(f_1, -f_2, -f_3)$ in the direction $\boldsymbol{e_1}$, and two rotlets of strength $\pm(\boldsymbol{f}\times\boldsymbol{e_1})$ at $\boldsymbol{x_{0,im}}$.

This requires the companion blob function for the potential dipole and the rotlets. More precisely, if ϕ_s is the blob function used to derive the Stokeslet at $\boldsymbol{x_0}$ then

$$\phi_d(r) = \frac{2}{r^5}\int_0^r t^4\phi_s(t)dt$$

where ϕ_d is the blob function used to derive the potential dipole and the rotlet of strength $-(\boldsymbol{f}\times\boldsymbol{e_1})$. With all these components, we get that $\boldsymbol{u}(\boldsymbol{x_e}) = \boldsymbol{0}$.

More generally, for $\boldsymbol{x} = (x_1, x_2, x_3)$ with $x_1 \geq w$, we have

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{S}_{\phi_s}[\boldsymbol{x}^*, \boldsymbol{f}] - \mathbf{S}_{\phi_s}[\widehat{\boldsymbol{x}}, \boldsymbol{f}] - \mathrm{h}^2\mathbf{PD}_{\phi_d}[\widehat{\boldsymbol{x}}, \boldsymbol{g}] + 2h\mathbf{SD}_{\phi_s}[\widehat{\boldsymbol{x}}, \boldsymbol{e_1}, \boldsymbol{g}]$$
$$+2h\mathbf{R}_{\phi_d}[\widehat{\boldsymbol{x}}, \boldsymbol{f}\times\boldsymbol{e_1}] - 2h\mathbf{R}_{\phi_s}[\widehat{\boldsymbol{x}}, \boldsymbol{f}\times\boldsymbol{e_1}]$$

where $\boldsymbol{x}^* = \boldsymbol{x}-\boldsymbol{x_0}$, $\widehat{\boldsymbol{x}} = \boldsymbol{x}-\boldsymbol{x_{0,im}}$, and $\boldsymbol{g} = 2(\boldsymbol{f}\cdot\boldsymbol{e_1})\boldsymbol{e_1}-\boldsymbol{f}$.

Finally, due to the linear relationship between force and velocity, we can form the matrix system:

$$\mathcal{U} = \mathcal{M}\mathcal{F}$$

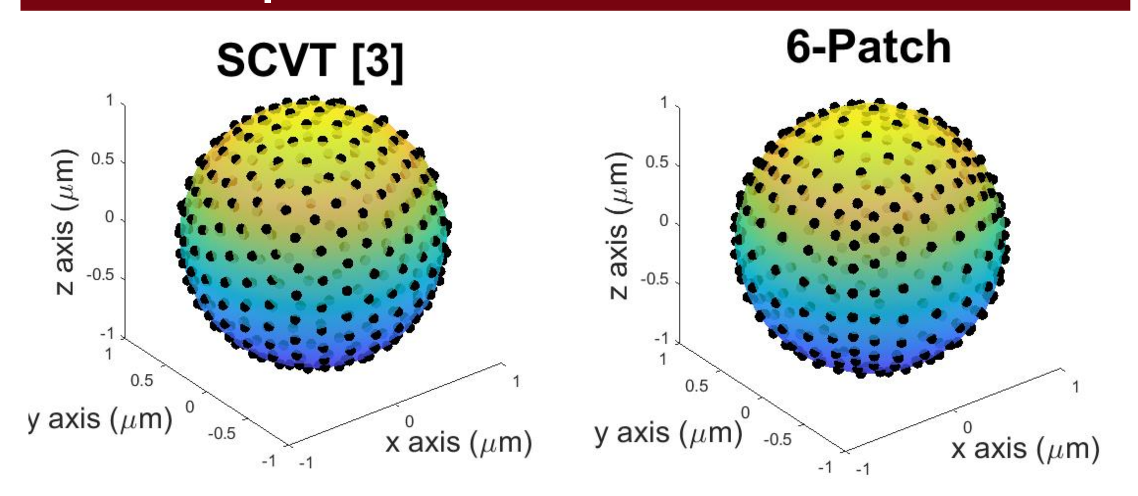
which allows us to impose a velocity on a sphere to solve for the net force on the sphere i.e.

$$\mathcal{M}^{-1}\mathcal{U} = \mathcal{F}.$$

Then the net torque is a sum of cross products of positions of discretized points on the sphere and the associated forces.

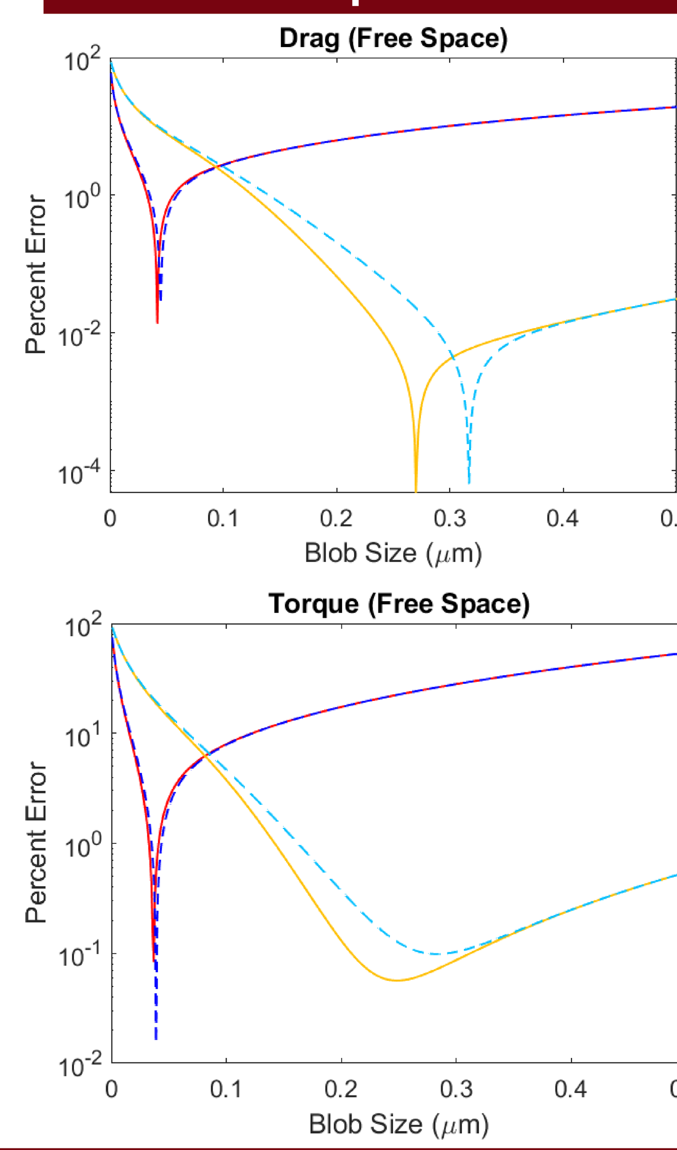
[2] Ainley J., Durkin S., Embid R., Boindala P., Cortez R, *The method of images for regularized Stokeslets*, Journal of Computational Physics, 227 (2008): 4600-4616.

Sphere Discretizations



[3] Du Q., Gunzburger M., Ju L., *Constrained centroidal Voronoi tessellations for surfaces*, SIAM J Sci Comput, 24 (2003):1488-1506.

Optimal Blob Sizes



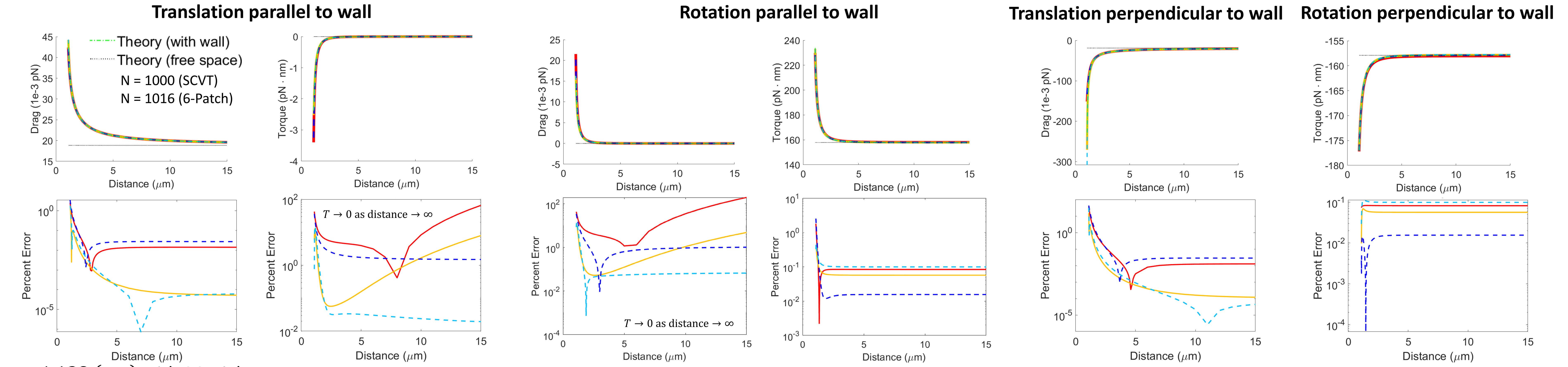
$$\text{Percent Error} = \frac{|S-T|}{|T|}\cdot 100$$

S is the simulate value, and T is the theoretical value.

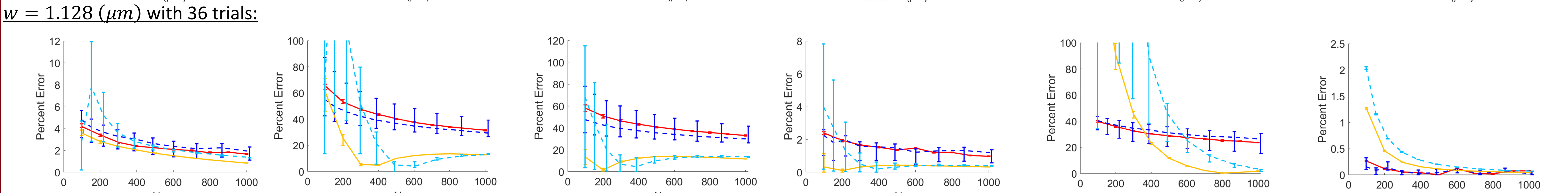
In free space, T is
$6\pi\mu R u$ for drag
$8\pi\mu R^3\Omega$ for torque
where:
μ is the fluid viscosity,
R the radius of the sphere,
u the sphere's velocity, and
Ω is the sphere's angular velocity.

Results

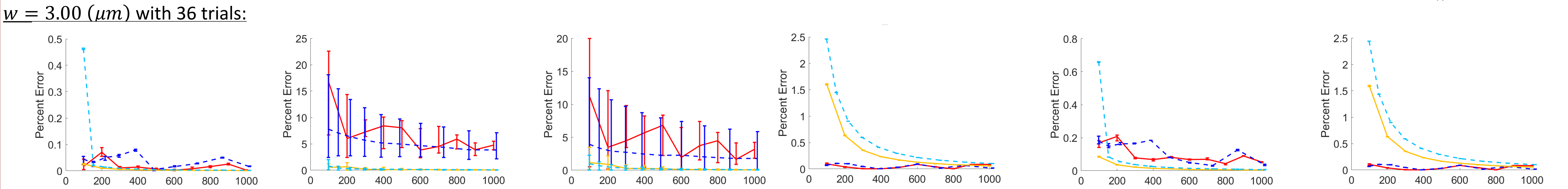
Translation parallel to wall | **Rotation parallel to wall** | **Translation perpendicular to wall** | **Rotation perpendicular to wall**



$w = 1.128$ (μm) with 36 trials:



$w = 3.00$ (μm) with 36 trials:

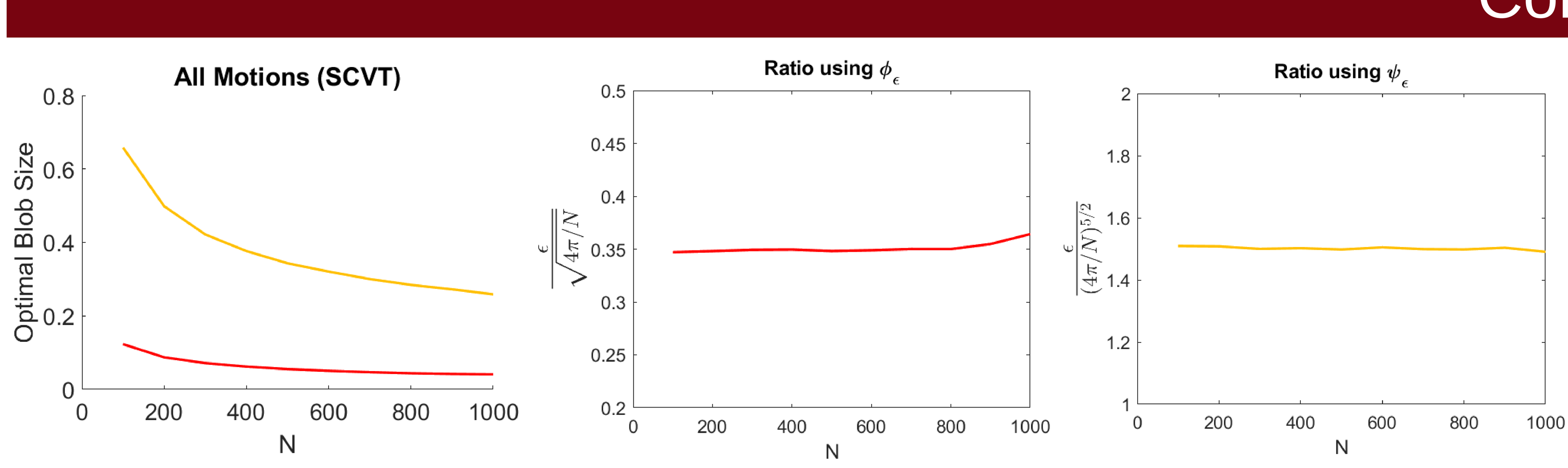


Distance is the measured from the center of the unit sphere to the wall

Percent error is calculated as specified in the "Optimal Blob Sizes" section except that the theoretical values are computed using [4].

[4] S. H. Lee, L. G. Leal, *Motion of a sphere in the presence of a plane interface. Part 2. An exact solution in bipolar co-ordinates*, Journal of Fluid Mechanics, 98 (1980): 193-224.

Conclusions



Blob function ψ_ϵ has the following advantages:

- More accurate results, particularly for discretization with a large number of points
- Smaller variance when the points of the discretization are perturbed

Discretization SCVT results are more accurate and have smaller variance because the point distribution is mostly uniform on the sphere surface and does not prefer any direction with respect to a boundary.

Optimal blob size for SCVT (in free space) is approximately:

$$\epsilon \sim \frac{7}{20}\sqrt{\frac{4\pi}{N}} \text{ for } \phi_\epsilon \text{ and } \epsilon \sim \frac{3}{2}\left(\frac{4\pi}{N}\right)^{2/5} \text{ for } \psi_\epsilon$$

Our study shows that SCVT using ψ_ϵ as the blob function is the most robust choice across all four spherical motions near a boundary.

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