A Combinatorial Proof of Schläfli's and Gould's Identity

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Outline

Stirling Numbers

Stirling Numbers of the 2nd Kind

Defintion

The **Stirling Numbers of the 2nd Kind** (OEIS A008277) is the sequence defined by

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

with $S(0,0) = 1$ and $S(n,k) = 0$ if $n, k < 0$ or $n > k$.

$S(n,k)$ counts the number of set partitions of $\{1,2,\ldots,n\}$ into k non-empty blocks.

$S(4,2) = 7$		
123/4	124/3	134/2
1/234	12/34	13/24
	14/23	

Stirling Numbers

Stirling Numbers of the 1st Kind

Definition

The **unsigned Stirling Numbers of the 1st kind** (OEIS A008275) is the sequence defined by

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$$

with the same initial conditions. Set $s(n,k) = (-1)^{n-k}c(n,k)$.

$c(n,k)$ counts the number of permutations of $\{1,2,\ldots,n\}$ with exactly k cycles.

$c(4,2)=11$			
(123)(4)	(132)(4)	(124)(3)	(142)(3)
(134)(2)	(143)(2)	(1)(234)	(1)(243)
(12)(34)	(13)(24)	(14)(23)	

Schläfli's and Gould's Identity

Schläfli's [Sch67] and Gould's Identity [Gou60]

Let $n, k \in \mathbb{Z}$ then

$$s(n, n-k) = \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} S(k+\alpha, \alpha)$$

and

$$S(n, n-k) = \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} s(k+\alpha, \alpha).$$

Stirling Permutations

Defintion

Gessel and Stanley first introduced **Stirling permutations** in [GS78]. They defined them to be permutation of the multiset $\{1, 1, 2, 2, \ldots, k, k\}$ which avoid 212, i.e. if $\pi = \pi_1 \ldots \pi_{2k}$ is a Stirling permutation then there is no subsequence of the form *bab* where $a < b$. Denote the set of all Stirling permutations as $\mathcal{Q}_k$.

Ex: $\pi = 1245543321 \in \mathcal{Q}_5$

Non-Ex: $1245533421 \notin \mathcal{Q}_5$

Stirling Permutations

Defintion

The number of **descents** in π, $\operatorname{des}\pi$, is the number of i such that $\pi_i > \pi_{i+1}$ or $i = 2k$.

Ex: $\pi = 1245543321 \in \mathcal{Q}_5$ **Ex:**
$\pi = 1245543321 \in \mathcal{Q}_5 \Rightarrow \operatorname{des}\pi = 5$ Gessel and Stanley proved that

$$S(k+n,n) = \sum_{\pi \in \mathcal{Q}_k} \binom{2k+n-\operatorname{des}\pi}{2k}$$

and

$$s(n,n-k) = (-1)^k \sum_{\pi \in \mathcal{Q}_k} \binom{n+\operatorname{des}\pi-1}{2k}.$$

An Observation

Note

$$s(n,n-k) = \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} S(k+\alpha,\alpha)$$

$$= \sum_{\pi \in \mathcal{Q}_k} \sum_{\alpha=\operatorname{des}\pi}^{k} (-1)^{\alpha} \binom{n+k}{k-\alpha} \binom{n+\alpha-1}{k+\alpha} \binom{2k+\alpha-\operatorname{des}\pi}{2k}$$

$$= \sum_{\pi \in \mathcal{Q}_k} (-1)^{k} \binom{n+\operatorname{des}\pi-1}{2k}.$$

The Identity

Theorem

For $0 \leq n, k, i$ and $j \in \mathbb{Z}$ with $i \leq k$,

$$\sum_{\alpha=i}^{k} (-1)^{\alpha} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{k+j+\alpha} \binom{2k-i+j+\alpha}{2k+j} = (-1)^{k} \binom{n+i-1}{2k+j}.$$

The Identity

Proof Sketch

$$\sum_{\alpha=i}^{k}(-1)^{\alpha}\binom{n+k}{n+\alpha}\binom{n+\alpha-1}{k+j+\alpha}\binom{2k-i+j+\alpha}{2k+j}$$

The sum is the weighted-count of triples (A, B, C) where

- $A \subseteq \{k-i+1, k-i+2, \ldots, k-i+n+k\}$ with $|A| = n+\alpha$,
- $B \subseteq A$ with $|B| = k+j+\alpha+1$ and B contains the largest element of A, and
- $C \subseteq \{1, 2, \ldots, k-i\} \cup B$ with $|C| = 2k+j+1$ and C contains the largest element of A

as α ranges from 0 to k and $\mathrm{wt}(A, B, C) = (-1)^{|A|-n}$.

The Identity

Proof Sketch

$$\sum_{\alpha=i}^{k}(-1)^{\alpha}\binom{n+k}{n+\alpha}\binom{n+\alpha-1}{k+j+\alpha}\binom{2k-i+j+\alpha}{2k+j}$$

Let $n=5,\ k=2,\ j=i=0$ and consider

$$(A,B,C)=(345789,3479,13479).$$

Set $d=\min(A^c\cup B\setminus C)=\min(6\cup\emptyset)=6\in A^c$ then

$$(A,B,C)\mapsto(3456789,4679,12479)=(A',B',C').$$

Note $d'=\min(A'^c\cup B'\setminus C')=\min(\emptyset\cup 6)=6\in B'\setminus C'$ so $(A',B',C')\mapsto(A,B,C)$ by deleting 6 from A',B'.

The Identity

Proof Sketch

$$\sum_{\alpha=i}^{k}(-1)^{\alpha}\binom{n+k}{n+\alpha}\binom{n+\alpha-1}{k+j+\alpha}\binom{2k-i+j+\alpha}{2k+j}$$

This map "fails" in three case:

1. $|A| = n+i$ and $d \in B \setminus C$,
2. $d \in A^c$ and $d > \max A$, or
3. no such d exists.

This leaves us with the sum

$$\sum_{\alpha=i}^{k}(-1)^{\alpha}\binom{n+\alpha-1}{k+j+\alpha}\binom{k-i}{k-\alpha} = (-1)^{k}\binom{n+i-1}{2k+j}.$$

Schläfli's and Gould's Identity Revisited

Theorem

For $0 \leq n, k, \gamma$,

$$\sum_{\alpha=0}^{k+\gamma} (-1)^{\alpha+\gamma} \binom{n+k+\gamma}{n+\alpha} \binom{n+\alpha-1}{k-\gamma+\alpha} S(k+\alpha, \alpha) = s(n, n-k)$$

and

$$\sum_{\alpha=0}^{k+\gamma} (-1)^{\alpha+\gamma} \binom{n+k+\gamma}{n+\alpha} \binom{n+\alpha-1}{k-\gamma+\alpha} s(k+\alpha, \alpha) = S(n, n-k).$$

General Sequences

Theorem

Let $k \geq 1$ and $A \in \mathbb{C}[t]$ with $0 \leq \deg A \leq k$. If

$$\sum_{n\geq 0} f(n)t^n = \frac{tA(t)}{(1-t)^{k+1}}$$

then

$$f(n) = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{\alpha} g(\alpha)$$

where

$$\sum_{n\geq 0} g(n)t^n = \frac{t^k A(1/t)}{(1-t)^{k+1}}.$$

An Application

Eulerian Polynomial

Set $A_k(t) = \sum_{\sigma \in S_k} t^{\operatorname{des} \sigma}$, then it is well-known that

$$\sum_{n \geq 0} n^k t^n = \frac{A_k(t)}{(1-t)^{k+1}}.$$

Moreover, due to the map $\sigma_1 \ldots \sigma_k \mapsto \sigma_k \ldots \sigma_1$, we get that $A_k(t) = t^{k+1} A_k(1/t)$. Thus, we get

$$n^k = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+k}{n+\alpha} \binom{n+\alpha-1}{\alpha} \alpha^k$$

which is known [Chu23].

An Application

Lah Numbers

The **Lah Numbers** (OEIS A271703), denoted $L(n,k)$, are

$$\frac{n!}{k!}\binom{n-1}{k-1}$$

and have the expontential generating function (for fixed $k \geq 1$)

$$\sum_{n\geq 0}\frac{L(n+k,k)}{(n+k)!}t^{n+1} = \frac{t/k!}{(1-t)^k}.$$

So, we have

$$\frac{L(n+k,k)}{(n+k)!} = \sum_{\alpha=1}^{k}(-1)^{\alpha+k}\binom{n+k}{n+\alpha}\binom{n+\alpha-1}{\alpha-1}\frac{L(\alpha,k)}{\alpha!}.$$

An Application

Narayana Numbers

The k-th **Narayana polynomial** is defined as

$$N_k(t) = \sum_{D \in \mathcal{D}_k} t^{\mathrm{peak}(D)} = \sum_{n \geq 0} N_{k,n} t^n$$

where $\mathcal{D}_k$ is the set of Dyck paths of semilength k. Then, for fixed $k \geq 1$,

$$\frac{N_k(t)}{(1-t)^{2k+1}} = \sum_{n \geq 0} N_{k+n,n} t^n.$$

So, we have

$$N_{k+n,n} = \sum_{\alpha=0}^{k} (-1)^{\alpha+k} \binom{n+2k}{n+k+\alpha} \binom{n+k+\alpha-1}{k+\alpha} N_{k+\alpha,\alpha}.$$

Thank You for Listening!
Questions?

`am3l14.github.io`

References

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